Robotics - Manipulator and Differential drive car

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1 Coordinate frames

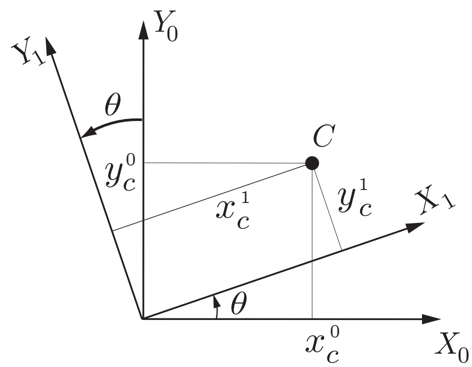

Figure 1: Two coordinate frames, Global reference frame, $O - X^0 - Y^0$ and local reference frame $O - X^1 - Y^1$. The latter is rotated by an angle θ from the former.

Coordinate frames are useful to represent rotational motion. The concepts in this part is core to robotics.

There is a global reference frame $O - X_0 - Y_0$. Consider another frame $O - X_1 - Y_1$ that is at an angle of θ to the global reference frame. Consider a random point C. This point has x-coordinate x_1^c along X_1 axis and y_1^c along Y_1 axis as shown. We are interested in finding the position in the global reference frame 0 or x_0^c and y_0^c.

We observe that $x_c^0 = x_c^1 \cos\theta - y_c^1 \sin\theta$ and $y_c^0 = x_c^1 \sin\theta + y_c^1 \cos\theta$. These two equations may be rewritten as

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} \tag{1}$$

2 Manipulator

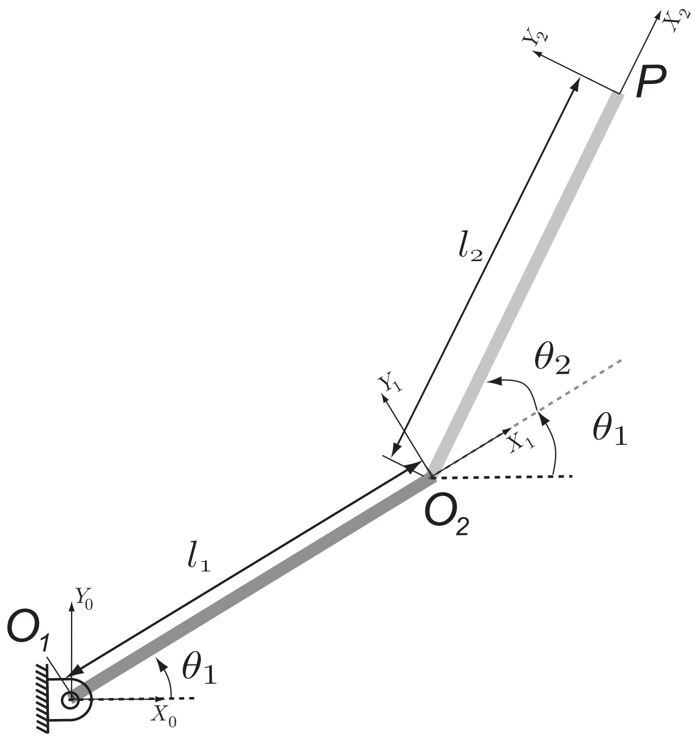

Figure 2: Double link manipulator

2.1 Forward kinematics

Consider the two-link manipulator as shown in Figure 2. The first link which is attached to a hinge at O_1 has a length of ℓ_1 and makes an angle of θ_1 with respect to the horizontal. The second link is attached to the first link at O_2, has a link length of ℓ_2 and makes an angle of θ_2 with the line extended from the first link as shown.

Forward kinematics refers to computing the positions of O_2 and P given the link lengths and angles. In order to do so, we attach a global reference frame $O_1 - X_0 - Y_0$ to the hinge joint. We attach local reference frames $O_2 - X_1 - Y_1$ to $P - X_2 - Y_2$ to point O_2 and P respectively.

The coordinates of point O_2 are obtained from Eqn. 1

$$\begin{bmatrix} x^0_{O_2} \\ y^0_{O_2} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \ell_1 \cos\theta_1 \\ \ell_1 \sin\theta_1 \end{bmatrix} \tag{2}$$

The coordinates of point P are obtained from Eqn. 1 and Eqn. 2

$$\begin{bmatrix} x_P^0 \\ y_P^0 \end{bmatrix} = \begin{bmatrix} x_{O_2}^0 \\ y_{O_2}^0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \ell_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \ell_1\cos\theta_1 + \ell_2\cos\theta_2 \\ \ell_1\sin\theta_1 + \ell_2\sin\theta_2 \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} x_P^0 \\ y_P^0 \end{bmatrix} = \begin{bmatrix} \ell_1\cos\theta_1 + \ell_2\cos\theta_2 \\ \ell_1\sin\theta_1 + \ell_2\sin\theta_2 \end{bmatrix} \tag{4}$$

In order to create an animation of the manipulator we use the MATLAB command *line* to join the points O_1 to O_2 and O_2 to P. Open and run the file *manipulator_forward.m*.

2.2 Inverse kinematics

Inverse kinematics is the problem of finding the joint angles for a given position of the end-effector. For instance, in the above example if we desire the end effector, the point P to be at $(x_P^0, y_P^0) = (x_{\text{ref}}, y_{\text{ref}})$ (fixed), then we need determine the angles θ_1 and θ_2.

Here we are trying to solve for θ_1 and θ_2 such that $x_P^0 = \ell_1\cos\theta_1 + \ell_2\cos\theta_2 = x_{\text{ref}}$ and $y_P^0 = \ell_1\sin\theta_1 + \ell_2\sin\theta_2 = y_{\text{ref}}$ This is a non-linear root finding problem. We use the MATLAB function *fsolve*. Open and run the file *manipulator_inverse.m*.

We can extend the above idea to draw a curve, the leminscate of figure of 8 curve. The equation of the lemniscate is $x_{\text{ref}} = \cos\phi$ and $y_{\text{ref}} = \cos\phi\sin\phi$ where $0 \le \phi \le 2\pi$. The key idea is to solve the inverse kinematics problem for multiple points to find the angles θ_1 and θ_2. Open and run the MATLAB file *manipulator_inverse_curve.m*.

3 Forward Kinematics of the differential Drive Car

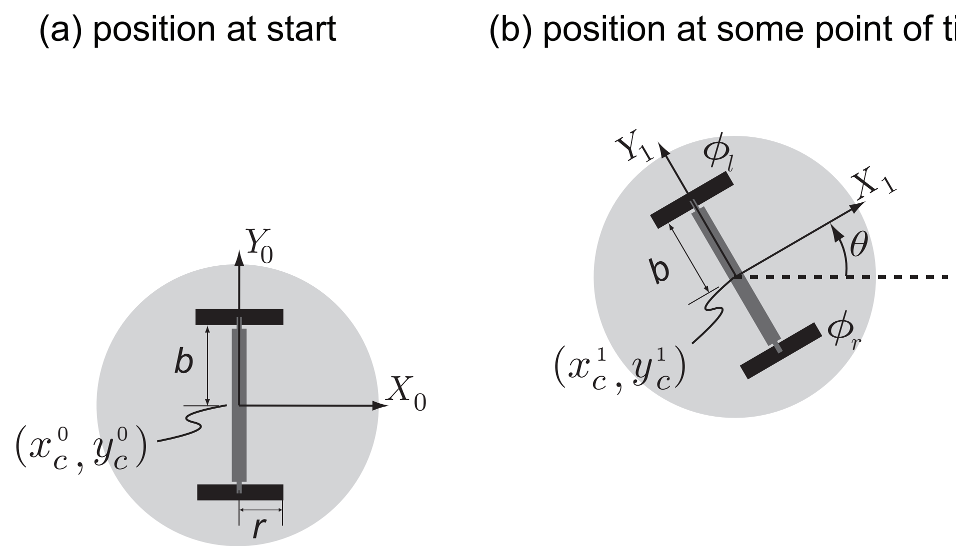

Figure 3: **Top view of differential drive car:** (a) The world or fixed frame $X_0 - Y_0$ is shown to coincide with robot local frame. (b) The local frame $X_1 - Y_1$ is attached to the robot and moves with it. The steer or heading angles is θ and the point C is midway between two wheels. It has the coordinates x_c^1, y_c^1 in the local frame and x_c^0, y_c^0 in the world or fixed frame. Here we find the differential equations for the position of C, $\dot{x}_c^0, \dot{y}_c^0$ and heading $\dot{\theta}$.

We use the Fig. 3 to derive an expression for the kinematics of the robot for the heading motion. We use two frames, $X_0 - Y_0$ is the fixed or world frame and $X_1 - Y_1$ is the local frame attached to the torso and moves as the torso moves. In this exposition, we are interested in keeping track of the mid-point on the torso (C), (x_c, y_c), and the heading angle θ.

We derive the velocity in the frame $O - X_1 - Y_1$. The x-component of the velocity $\dot{x}_c^1$ is given by the average speed of the wheels while the y-component of the velocity $\dot{y}_c^1$ is zero because the y-axis is always normal to the heading direction in the local frame. Thus

$$\begin{aligned}\dot{x}_c^1 &= 0.5r(\dot{\phi}_r + \dot{\phi}_l)\\ \dot{y}_c^1 &= 0\end{aligned}$$

The x- and y- components of the velocity in frame $O - X_0 - Y_0$ can be rewritten as

$$\begin{aligned}\begin{bmatrix}\dot{x}_c^0\\ \dot{y}_c^0\end{bmatrix} &= \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\dot{x}_c^1\\ \dot{y}_c^1\end{bmatrix}\\ &= \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}0.5r(\dot{\phi}_r + \dot{\phi}_l)\\ 0\end{bmatrix}\end{aligned}$$

The above expressions are simplified to

$$\dot{x}_c^0 = 0.5r(\dot{\phi}_r + \dot{\phi}_l)\cos(\theta)$$
$$\dot{y}_c^0 = 0.5r(\dot{\phi}_r + \dot{\phi}_l)\sin(\theta)$$

The angular velocity for heading $\dot{\theta}$ may be obtained arguing about the change in heading as the speed of the two wheels changes. For instance, if the left wheel is stationary the car will start turning with a radius $2b$. The arc length made by the right wheel is $r\dot{\phi}$. Thus, $\dot{\theta} = r\dot{\theta}/2b$. Repeating the same argument for the other wheel we have

$$\dot{\theta} = 0.5\frac{r}{b}(\dot{\phi}_r - \dot{\phi}_l)$$

To simplify interpretation, we substitute $\omega = \dot{\phi}_r + \dot{\phi}_l$ which is the average speed and $\Omega = \dot{\phi}_r + \dot{\phi}_l$ which is the net rotation of the car. Thus, our equations are

$$\dot{x}_c^0 = 0.5r\omega\cos(\theta) \tag{5}$$
$$\dot{y}_c^0 = 0.5r\omega\sin(\theta) \tag{6}$$
$$\dot{\theta} = 0.5\frac{r}{b}\Omega \tag{7}$$

Unlike the manipulator, the forward kinematics of the differential drive car is a differential equation. In order to animate the system, we will need to find the position $x_c^0(t_i)$, $y_c^0(t_i)$, and $\theta(t_i)$ where t_i is the time step $i = 0, 1, 2, ...$ We do this by integrating the equations of motion using Euler's method. Assume that h is the time step and is given, the known initial states of $x_c^0(t_0)$, $y_c^0(t_0)$, and $\theta(t_0)$. Then we can find the position and heading using the formula below.

$$x_c^0(t_{i+1}) = x_c^0(t_i) + 0.5r\omega(t_i)\cos\theta(t_i)h$$
$$y_c^0(t_{i+1}) = y_c^0(t_i) + 0.5r\omega(t_i)\sin\theta(t_i)h$$
$$\theta(t_{i+1}) = \theta(t_i) + 0.5\frac{r}{b}\Omega(t_i)h$$

Note that the input to the system is the speed $\omega(t_i)$ and heading direction $\Omega(t_i)$. Open and run the MATLAB file *diff_drive_main.m*