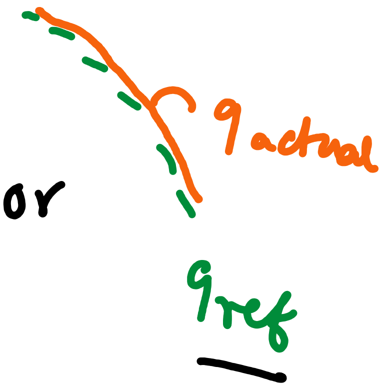
Trajectory Tracking control

Equations of motion

Euler-Lagrange for manipulator

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j}$$



$\rightarrow$ $M(q)\,\ddot{q} + C(q,\dot{q}) + G(q) = \tau$

$M(q)$ – mass matrix

$C(q,\dot{q})$ – coriolis acceleration

$G(q)$ – gravitional acceleration

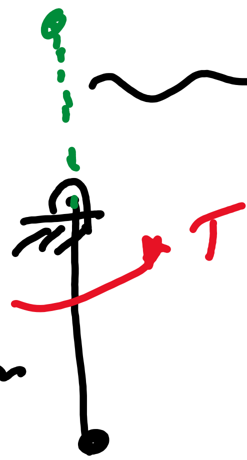
τ – external torque

$$A\,\ddot{q} = b$$

$$M(q)\,\ddot{q} = (\tau - C(q,\dot{q})\dot{q} - G(q))$$

Two objectives of control

① Set-point control



upright $\theta = 0$

set-point

hanging down

benchmark problem

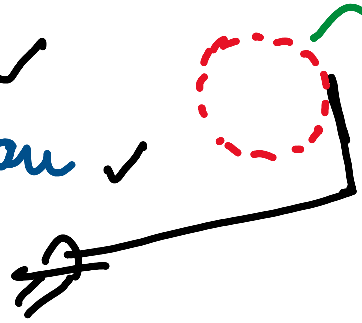
balance a stick.

② Trajectory tracking control

Two spaces

i) Joint ✓

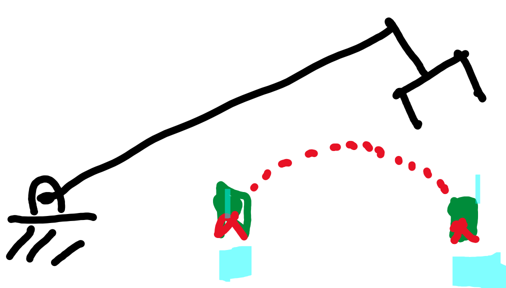
ii) Cartesian ✓



track a curve

We will focus on this.

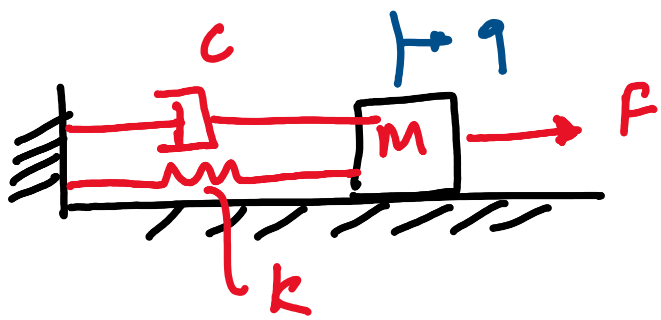
① is a special case of ②



Simple system with equations similar to a manipulator

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

$$\boxed{m\ddot{q} + c\dot{q} + kq = F}$$ (spring mass damper)



lets assume $F = 0$ (free vibration)

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = 0$$

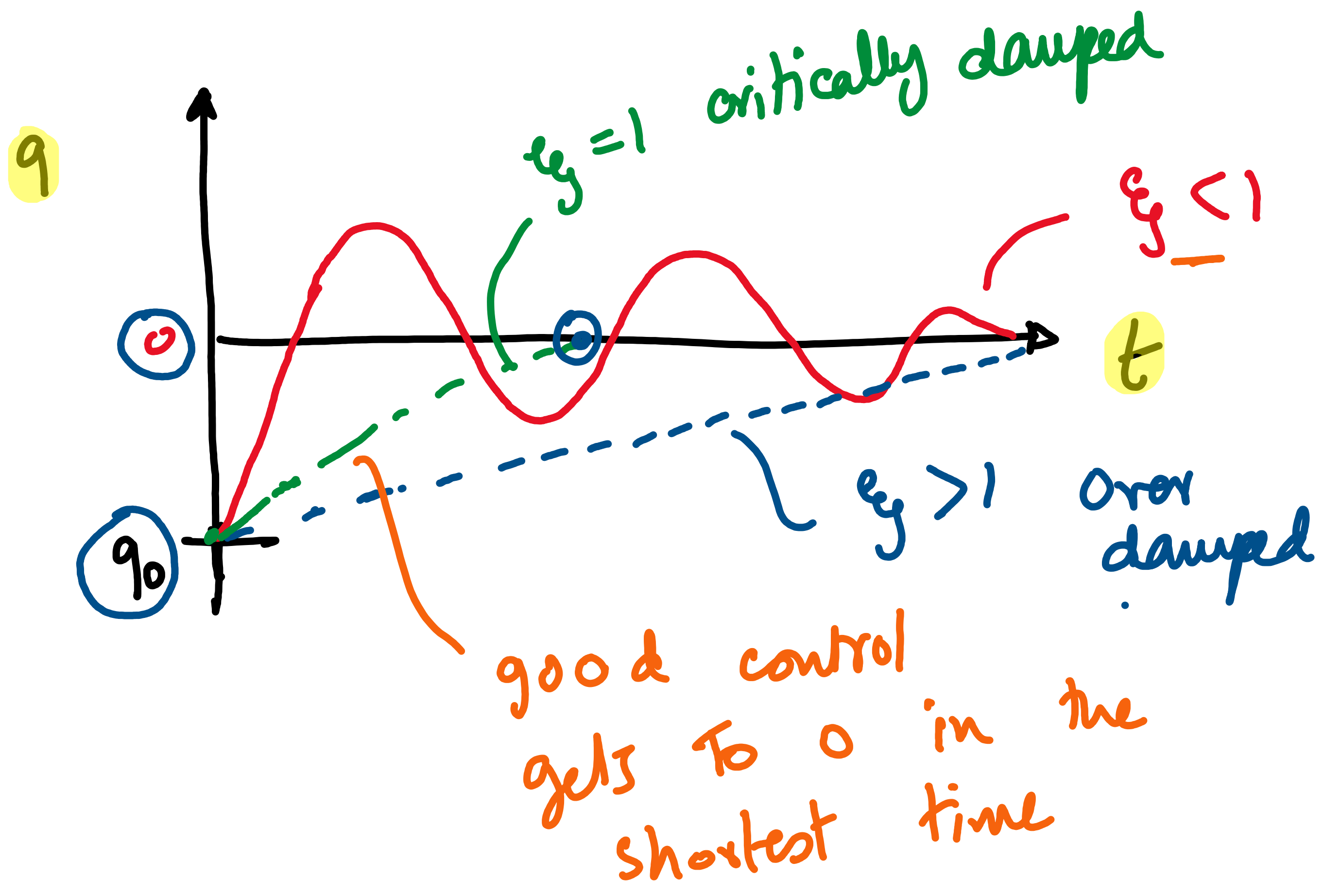
$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \qquad 2\xi\omega_n = \frac{c}{m}$$

Xi

$$\boxed{\xi = \frac{c}{2\sqrt{km}}}$$

3 cases

1. $\boxed{\xi > 1}$ $\quad c > 2\sqrt{km}$ Over damped
2. $\boxed{\xi = 1}$ $\quad c = 2\sqrt{km}$ Critical damped
3. $\boxed{\xi < 1}$ $\quad c < 2\sqrt{km}$ Under-damped



$$m\ddot{q} + c\dot{q} + kq = F \qquad -(1)$$

Design F such that the system is critically damped.

Assume $$F = -k_p q - k_d \dot{q} \qquad -(2)$$

proportional - derivative control

Sensors : $q, \dot{q}$

Substitute (2) in (1)

$$m\ddot{q} + c\dot{q} + kq = \left(-k_p q - k_d \dot{q}\right) = F$$

$$m\ddot{q} + (c + k_d)\dot{q} + (k + k_p)q = 0$$

free vibrations

Choose k_p, k_d such that the system is critically damped

$\xi = 1$

$c = 2\sqrt{km}$

$$\boxed{(c + k_d) = 2\sqrt{(k + k_p)\,m}}$$

2 constants and 1 equation.
Fix one & use the equation to compute the second one.

Fix k_p, solve for k_d

$$c + k_d = 2\sqrt{(k+k_p)m}$$

Square

$$(c+k_d)^2 = 4\,(mk + mk_p)$$

$$k_d^2 - 2ck_d + c^2 - 4mk - 4mk_p = 0$$

2 roots, choose the positive root.

$$\boxed{k_d = -c + 2\sqrt{(k+k_p)m}}$$

system will be critically damped.

Extend the idea to 2D

1D: $m\ddot{q} + c\dot{q} + kq = F$ ✓

2D:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$F = -k_p q - k_d \dot{q}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -\underbrace{\begin{bmatrix} k_{p11} & k_{p12} \\ k_{p21} & k_{p22} \end{bmatrix}}_{\text{4 parameters}}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \underbrace{\begin{bmatrix} k_{d11} & k_{d12} \\ k_{d21} & k_{d22} \end{bmatrix}}_{\text{4 parameters}}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

8 parameters

To have a critical damped system ξ_1, ξ_2 there will be 2 equations.

Free parameters = $8 - 2 = 6$ (Too many)

Inverse Dynamic Control (IDC)

Dynamics : $M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$ ①

Goal : Track a reference $q_r, \dot{q}_r, \ddot{q}_r$

IDC : $\tau = M(q)\left[\ddot{q}_r + K_d(\dot{q}_r - \dot{q}) + K_p(q_r - q)\right] + C(q,\dot{q}) + G(q)$ ②

Substitute ② in ①

$$\underbrace{M(q)\ddot{q} + \cancel{C(q,\dot{q})} + \cancel{G(q)}}_{①} = \underbrace{M(q)\left[\ddot{q}_r + K_d(\dot{q}_r - \dot{q}) + K_p(q_r - q)\right] + \cancel{C(q,\dot{q})} + \cancel{G(q)}}_{②}$$

$$M(q)\left[(\ddot{q}_r - \ddot{q}) + K_d(\dot{q}_r - \dot{q}) + K_p(q_r - q)\right] = 0$$

$$M(q)\left[\ddot{e} + K_d\dot{e} + K_p e\right] = 0$$

where $e = \underbrace{q_r}_{\text{reference}} - \underbrace{q}_{\text{actual joint angle}}$

Since $M(q) \neq 0 \Rightarrow \ddot{e} + k_d \dot{e} + k_p e = 0$

But these are n-decoupled equations

$$\begin{bmatrix} \ddot{e}_1 \\ \ddot{e}_2 \\ \vdots \\ \ddot{e}_n \end{bmatrix} + \begin{bmatrix} k_{d_1} & 0 & \cdots & 0 \\ 0 & k_{d_2} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & k_{d_n} \end{bmatrix} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_n \end{bmatrix} + \begin{bmatrix} k_{p_1} & 0 & 0 & 0 \\ 0 & k_{p_2} & 0 & 0 \\ 0 & & & \\ 0 & 0 & \cdots & k_{p_n} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = 0$$

$$\left.\begin{array}{ll} q_{1r} - q_1 & \ddot{e}_1 + k_{d_1} \dot{e}_1 + k_{p_1} e_1 = 0 \\ q_{2r} - q_2 & \ddot{e}_2 + k_{d_2} \dot{e}_2 + k_{p_2} e_2 = 0 \\ & \vdots \\ q_{nr} - q_n & \ddot{e}_n + k_{d_n} \dot{e}_n + k_{p_n} e_n = 0 \end{array}\right\} \text{n decoupled equations}$$

$$\ddot{e}_i + k_{d_i}\dot{e}_i + k_{p_i}e_i = 0 \quad \leftarrow i = 1, \ldots n$$

Compare against

$$\rightarrow \quad m\ddot{q} + (k_d + c)\dot{q} + (k_p + k)q = 0$$

$$m = 1 \; ; \quad c = 0 \; ; \quad k = 0$$

For critical damping

$$k_d = -c + 2\sqrt{(k + k_p)m}$$

Substitute $m = 1$ $\quad c = k = 0$

$$\rightarrow \quad \boxed{k_{d_i} = 2\sqrt{k_{p_i}}} \quad i = 1, 2, \ldots n$$

Inverse dynamics control

$$\rightarrow \tau = M(q)\left[\ddot{q}_r + K_d(\dot{q}_r - q) + K_p(q_r - q)\right] + C(q,\dot{q}) + G(q)$$

$M(q)$, $C(q,\dot{q})$, $G(q)$ need sensor measurements

$q, \dot{q}$ joint position, velocity

sensor measurements

- are noisy *
- delayed. *

Feedforward and feedback

This control replace q with q_r in $M(q)$, $C(q,\dot{q})$, $G(q)$

$$\tau = M(q_r)\left[\ddot{q}_r + K_d(\dot{q}_r - q) + K_p(q_r - q)\right] + C(q_r,\dot{q}_r) + G(q_r)$$

sensor

Use reference $q_r, \dot{q}_r$

e.g. quintic polynomial

Lyapunov's Direct Method (prove stability)

Need to find an energy like function $V(x)$ that decreases over time. If such a function is found it shows that the system is stable

Consider the system $\dot{x} = f(x)$ with equilibrium point ($\dot{x} = 0$) at $x = 0$
i.e. $f(0) = 0$

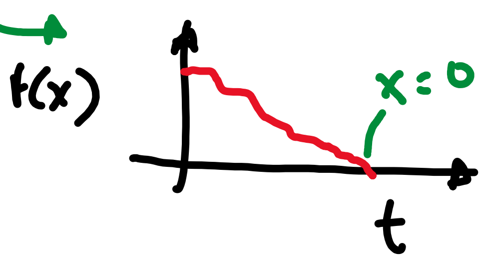
$V(x)$ is a Lyapunov function if

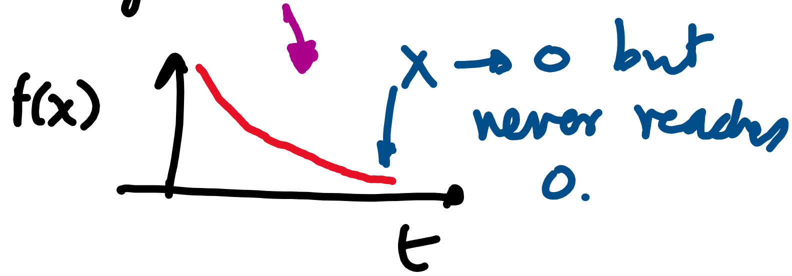
1. $V(x) > 0 \quad x \neq 0$ Positive definite

2. $\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx} f(x) \leq 0$ Negative definite

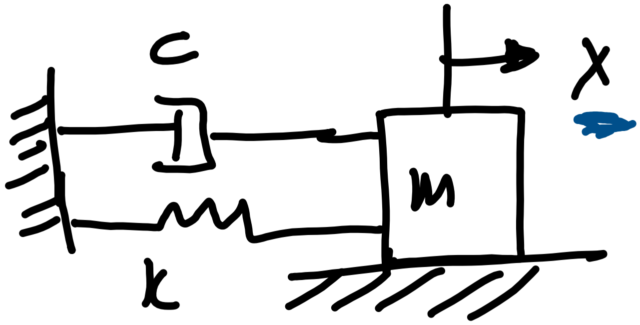
then $x = 0$ is stable

However if $V(x) > 0$ and $\dot{V}(x) < 0$ then $x = 0$ is asymptotically stable





EXAMPLE



Equation of motion $m\ddot{x} + c\dot{x} + kx = 0$

$c > 0$

The equilibrium point is $x = 0$

Choose the Lyapunov function

$\longrightarrow \quad V(x) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$ (energy)

such that $V(x) > 0 \quad x, \dot{x} \neq 0$ (1)

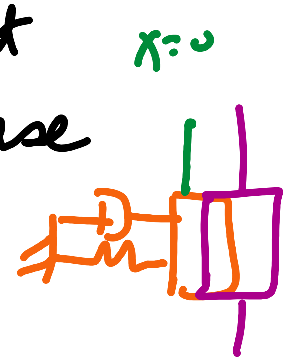
$$\dot{V}(x) = m\dot{x}\ddot{x} + kx\dot{x}$$
$$= \dot{x}(m\ddot{x} + kx)$$
$$= \dot{x}(-c\dot{x})$$
$$\dot{V}(x) = -c\dot{x}^2 \quad (2) \quad c > 0$$

$$\{m\ddot{x} + kx = -c\dot{x}\}$$

This shows that $\dot{V}(x)$ is decreasing (since $c > 0$) as long as $\dot{x} \neq 0$.

$\dot{V}(x) = -c\dot{x}^2$

But there is a possibility that $\dot{x} = 0$ at $x \neq 0$ in which case stability is not proven since $\dot{V}(x) = 0$ at $x \neq 0$.



$x=0$ where $\dot{x}=0$

→ La Salle's Invariance Principle

For the system $\dot{x} = f(x)$ with a Lyapunov function $V(x) > 0$ & $\dot{V}(x) \leq 0$

If $x=0$ (equilibrium) is the lone point such that $\dot{V}(x) = 0$ then $x=0$ is asymptotically stable.

Going back to the spring-mass-damper

$$\dot{V}(x) = -c\dot{x}^2 = 0 \quad \text{at} \quad \dot{x} = 0 \leftarrow \cancel{x=0}$$

$\ddot{x} = 0$

This implies that $\ddot{x} = 0$

Substitute in the equations of motion

$$m\underset{\ddot{x}=0}{(0)} + c\underset{\dot{x}=0}{(0)} + kx = 0 \Rightarrow x = 0$$

→ $x=0$ is the lone point st $\dot{V}(x)=0$

$x=0$ is asymptotically stable.

Proportional - Derivative Controller

$$\tau = -K_p \underset{e}{q} - K_d \underset{\dot e}{\dot q}$$

Equilibrium $q=0$

If there was a ref.

$q_r \neq 0 \quad e = q - q_r$

$\boxed{\text{Case 1}}$: No gravity $[G(q)=0]$

$$M\ddot q + C(q,\dot q) = \tau$$

$$M\ddot q + C(q,\dot q) + \underbrace{K_d \dot q + K_p q}_{} = 0 \quad \text{— (1)}$$

Lyapunov function

(1) $$V(q) = \frac{1}{2}\dot q^T M \dot q + \frac{1}{2} q^T K_p q \quad ✓$$

$$\dot V(q) = \dot q^T M \ddot q + 0.5\, \dot q^T \dot M \dot q + \dot q^T K_p q$$

$$= \dot q^T [M\ddot q + 0.5 \dot M \dot q + K_p q]$$

$$= \dot q^T [0.5 \dot M \dot q - \underbrace{C(q,\dot q) - K_d \dot q}_{\text{From (1)}}]$$

0''

It can be shown that

$$0.5 \dot M \dot q - C(q,\dot q) = [0.5\dot M - \bar C(q,\dot q)]\dot q$$

and $0.5 \dot M - \bar C(q,\dot q)$ is skew symmetric matrix.

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

The term $\dot{q}^T[0.5\dot{M} - \bar{C}(q,\dot{q})]\dot{q} = 0$ because $0.5\dot{M} - \bar{C}(q,\dot{q})$ is skew symmetric matrix

(2) $\boxed{\dot{V}(q) = -\dot{q}^T K_d \dot{q} \leq 0}$

Although $\dot{V}(q)$ is decreasing, it is possible that $\dot{q} = 0$ at $q \neq 0$.

We can now use La Salle's invariance

If $\dot{q} = 0 \Rightarrow \ddot{q} = 0$

Substitute in

$$M\cancel{\ddot{q}} + \bar{C}(q,\dot{q})\cancel{\dot{q}} + K_d\cancel{\dot{q}} + K_p q = 0$$

Thus $q = 0$ is the lone equilibrium point when $\dot{V}(q) = 0$

Hence a PD controller leads to asymptotic stability for no gravity case.

Case 2: Gravity case

$$M\ddot{q} + C(q,\dot{q}) + \boxed{G(q)} = \tau$$

$$M\ddot{q} + C(q,\dot{q}) + G(q) + K_d\dot{q} + K_p q = 0 \quad (1)$$

When the system reaches steady state

$\dot{q} = 0 \Rightarrow \ddot{q} = 0$

From (1) $\quad G(q) + K_p q = 0$

$$\boxed{K_p q = -G(q)} \quad q \neq 0$$

steady state error

this implies that $q \neq 0$. One can make q small by increasing K_p to a large value but $q \neq 0$.

Thus, a PD controller cannot achieve a steady state error of zero.

This can be fixed in two ways

(i) Add gravity compensation

(ii) Add integral control.

(i) Gravity compensation

$$\boxed{\tau = G(q) - K_p q - K_d \dot{q}}$$

Substitute into $\underline{M\ddot{q} + C(q,\dot{q}) + G(q)} = \tau$

$$M\ddot{q} + C(q,\dot{q}) + \cancel{G(q)} = \cancel{G(q)} - K_p q - K_d \dot{q}$$

$$M\underbrace{\ddot{q}}_{=0} + \underbrace{C(q,\dot{q})}_{=0} + K_p q + K_d \dot{q} = 0$$

When $\dot{q} = 0$, $\ddot{q} = 0$, implies $K_p q = 0$, $q = 0$

(ii) Proportional-Integral-Derivative Control

$$\boxed{\tau = -K_p q - K_d \dot{q} - K_I \int q \, dt}$$

Substitute in equation: $M\ddot{q} + C(q,\dot{q}) + G(q) = \tau$

$$M\underbrace{\ddot{q}}_{=0} + \underbrace{C(q,\dot{q})}_{=0} + G(q) = -K_p q - \underbrace{K_d \dot{q}}_{=0} - K_I \int q \, dt$$

At $\dot{q} = 0 \Rightarrow \ddot{q} = 0$

$$K_p q + K_I \int q \, dt = -G(q)$$

Taking derivative wrt time

$$K_p \underbrace{\dot{q}}_{=0} + K_I q = \underbrace{0}_{\frac{\partial G}{\partial q}\underbrace{\dot{q}}_{=0}} \Rightarrow q = 0$$

Summary

Manipulator : $M(q) + C(q,\dot{q}) + G(q) = \tau$

① Proportional - Derivative Control (PD control)

$$\tau = -k_p(q - q_r) - k_d(\dot{q} - \dot{q}_r)$$

Use for slow speed and no gravity conditions

② Gravity + PD control

$$\tau = \underline{G(q)} - k_p\underline{(q - q_r)} - k_d\underline{(\dot{q} - \dot{q}_r)}$$

q = sensor

If sensor measurements are delayed then replace $G(q)$ with $G(q_r)$ q_r = reference

③ Proportional - Integral - Derivative Control

$$\tau = -k_p(q - q_r) - k_d(\dot{q} - \dot{q}_r) - k_I \sum (q - q_r)$$

Use when the model parameters M, C, G are uncertain or unknown. The I term helps to cancel constant disturbance.

④ Inverse Dynamics Control

$$\tau = G(q) + C(q,\dot{q}) + M(q)\left[\ddot{q}_r + K_p(q_r - q) + K_d(\dot{q}_r - \dot{q})\right]$$

Use when
- i) accurate model is available
- ii) less noisy sensors
- iii) no sensor delay

M, C, G.

q estimate is good / no delay

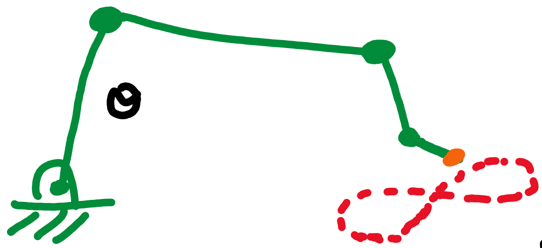
⑤ Feed forward - Feedback Control

$$\tau = G(q_r) + C(q_r,\dot{q}_r) + M(q_r)\left[\ddot{q}_r + K_p(q_r - q) + K_d(\dot{q}_r - \dot{q})\right]$$

sensors

Use when
- i) accurate model is available
- ii) noisy sensors ✓
- iii) time delayed sensor measurements ✓

Task Space control



given

$x_r \,,\; q_r$

position, quaternion

Given $\underline{x_r}, \underline{\dot{x}_r}, \underline{\ddot{x}_r}, q_r, \dot{q}_r, \ddot{q}_r$

Convert $\dot{q}_r, \ddot{q}_r$ to $\omega_b, \dot{\omega}_b$

$$\omega_b = 2\,\underline{\dot{q}_r} \cdot \underline{\bar{q}_r} \quad \text{and} \quad \dot{\omega}_b = 2\,\underline{\ddot{q}_r} \cdot \bar{q}_r + 2|\dot{q}_r|^2$$

Transform from cartesian to joint space

$$\underline{\theta} = \underline{FK}^{-1}(x_r, q_r)$$

$$\underline{\dot{\theta}} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}^{-1} \begin{bmatrix} \dot{x}_r \\ \omega_b \end{bmatrix}$$

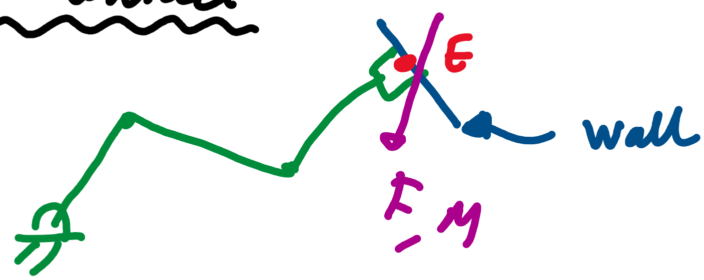
finite difference

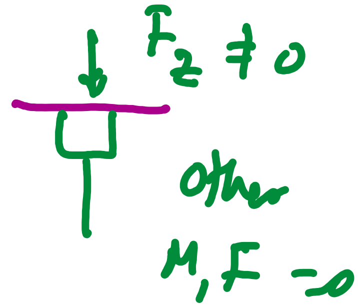
$$\underline{\ddot{\theta}} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \ddot{x}_r \\ \dot{\omega}_b \end{bmatrix} - \begin{bmatrix} \dot{J}_v \\ \dot{J}_\omega \end{bmatrix} \dot{\theta} \right\}$$

$\dot{J} = 0$

$$J\dot{\theta} = \begin{pmatrix} \dot{x}_r \\ \omega_b \end{pmatrix} \Rightarrow \underbrace{J}_{J^{-1}}\ddot{\theta} + \dot{J}\dot{\theta} = \begin{pmatrix} \ddot{x}_r \\ \dot{\omega}_b \end{pmatrix}$$

Manipulator Contact





$$\rightarrow M(q)\ddot{q} + C(q,\dot{q}) + G(q) + J_E^T \begin{bmatrix} F \\ M \end{bmatrix} = \tau$$

$$J_E = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \quad \text{at tip}$$

F, M = force / moment at tip due to contact
[not all 6 force/moment need to be present]

Force control:

$$\tau = M(q_r)\ddot{q}_r + C(q_r, \dot{q}_r) + G(q_r) + J_E^T \begin{bmatrix} F_r \\ M_r \end{bmatrix}$$

F_r, M_r is the reference force at the tip E.

NOTE: If some of these forces/moment references are zero, then just replace those with zeros.

Incase there is a force/moment sensor that can measure end-effetor force then

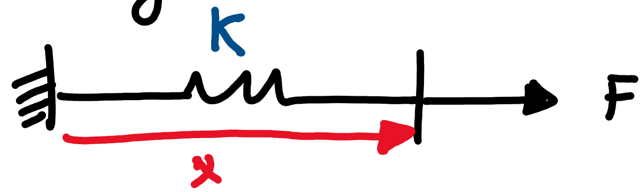
$$\tau = M\ddot{q}_r + C(q_r, \dot{q}_r) + G(q_r) + J_E^T \begin{bmatrix} \overbrace{F_r - F}^{\Delta F} \\ \underbrace{\dot{M}_r - M}_{\Delta M} \end{bmatrix}$$

F, M are the measured force/moments.

Impedance Control (Task space control)

Impedance is approximately the stiffness (K)

In 1D: $K = \frac{F}{x}$



Here x is the input and F is the output. Impedance control achieves cartesian space control in a soft way as follows

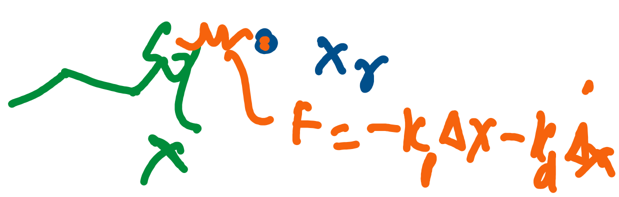
$$\tau = M\ddot{q}_r + C(q_r, \dot{q}_r) + G(q_r) + J_E^T \begin{bmatrix} F_e \\ M_e \end{bmatrix}$$

$$\begin{bmatrix} F_e \\ M_e \end{bmatrix} = K_p \begin{bmatrix} x_r - x \\ \underbrace{quat_r - quat}_{\text{non-constant quat}} \end{bmatrix} + K_v \begin{bmatrix} \dot{x}_r - \dot{x} \\ \omega_r - \omega \end{bmatrix}$$

The net result is that $x_r = x$ and $quat_r = quat$.

This type of control is used when tracking is desired but the end-effector may also make contact with the environment.

$\Delta x = x - x_r \Rightarrow 0$



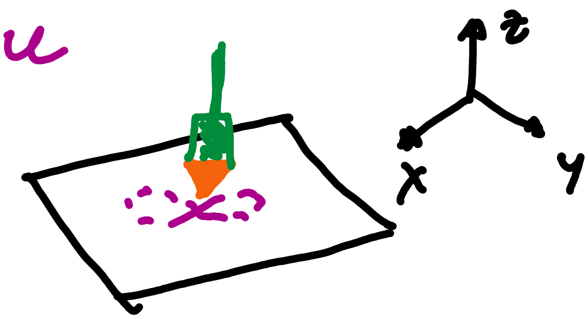
Hybrid Force / Position Control

This is useful when the end-effector has to apply forces in some directions and move in some other directions.

For example, a robot with a polishing end tool needs to apply a force in a direction normal to the surface but it also needs to move in direction parallel to the surface

→ F_z apply force

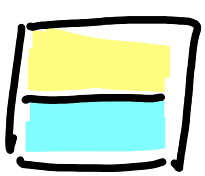
→ x-y move



Hybrid

F-X

$$\tau = M\ddot{q}_r + C(q_r, \dot{q}_r) + G(q_r) + J_E^T \begin{bmatrix} -K_p \Delta x - K_d \Delta \dot{x} \\ -K_p \Delta y - K_d \Delta \dot{y} \\ F_z \end{bmatrix}$$



Position Control

Force Control

Hybrid.