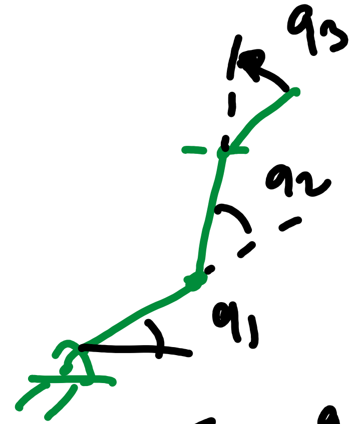
Trajectory generation

① Joint space:



$q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$

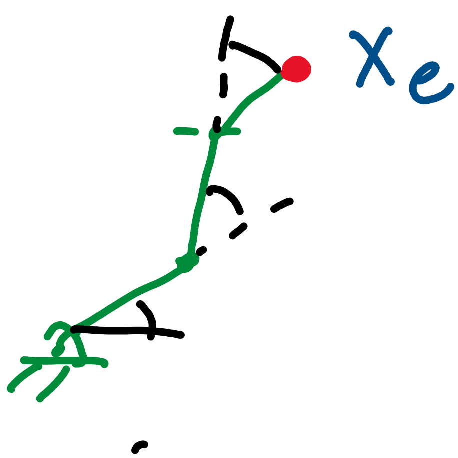
$q \rightarrow$ joint angles

$q = [q_1, q_2, q_3]$

② Task space

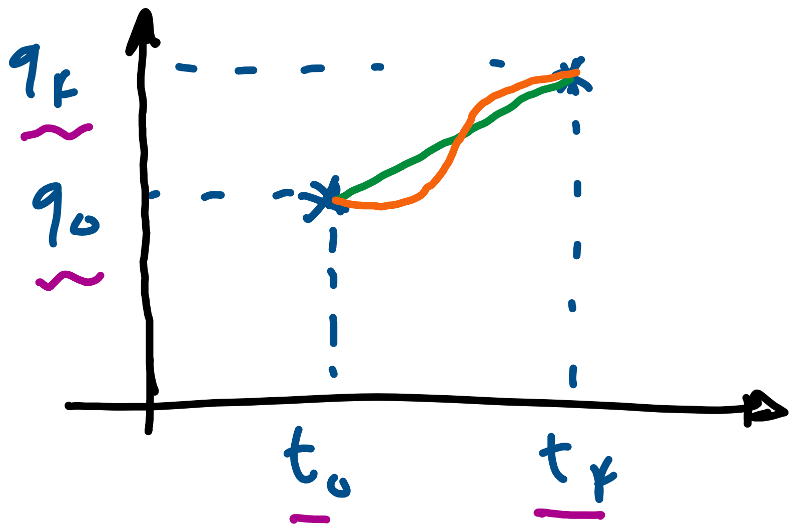
$x_e(t)$, $\dot{x}_e(t)$, $\ddot{x}_e(t)$

$x_e \rightarrow$ end-effector position/orientation



① Joint space

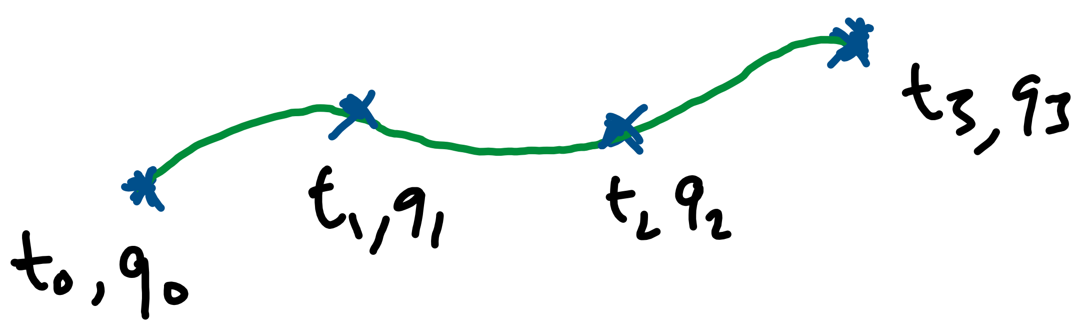
(i) Point-to-point



q - joint angle

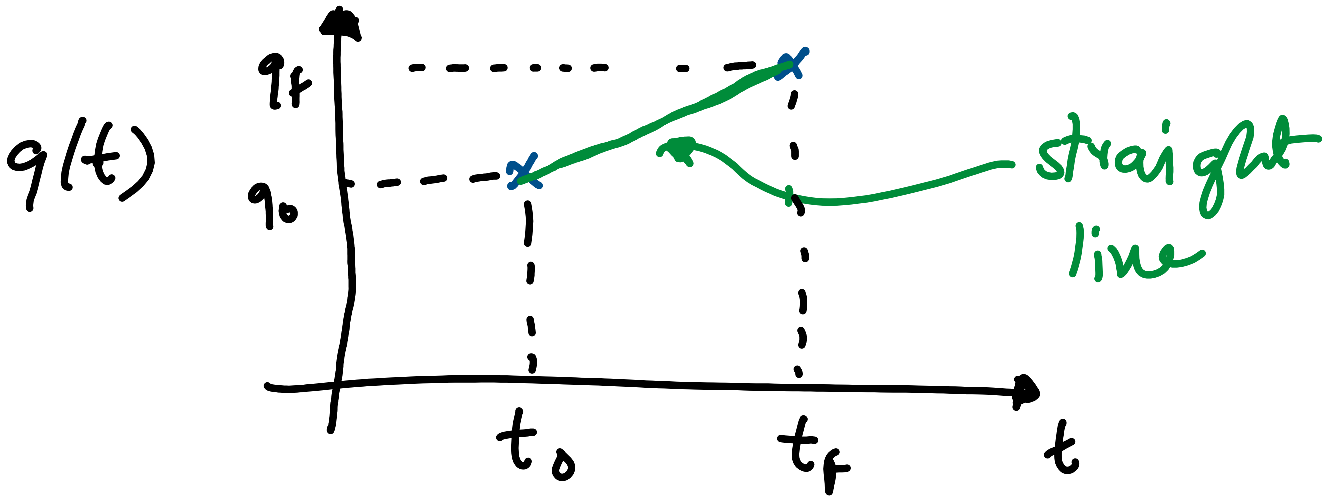
linear, quadratic, quintic polynomials

(ii) Via points



piecewise cubic spline

① Linear profile



$$\rightarrow q(t) = a_0 + a_1 t$$

a_0, a_1 unknown constants

$q(t_0) = q_0 \Rightarrow q_0 = a_0 + a_1 t_0$

$q(t_f) = q_f \Rightarrow q_f = a_0 + a_1 t_f$

2 eq$^{\underline{n}}$s

2 unknowns

a_0, a_1

$$\underbrace{\begin{bmatrix} q_0 \\ q_f \end{bmatrix}}_{b\,\checkmark} = \underbrace{\begin{bmatrix} 1 & t_0 \\ 1 & t_f \end{bmatrix}}_{A\,\checkmark} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{X}$$

$$\overset{\checkmark}{b} = \overset{\checkmark}{A} \underset{?}{X}$$

$$\underline{x} = A^{-1} b$$

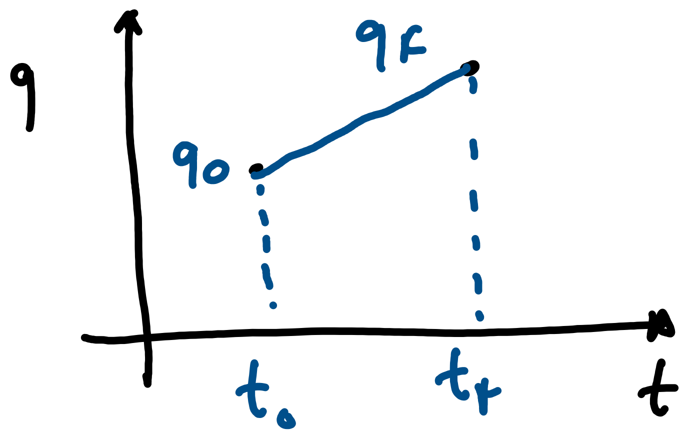
$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & t_0 \\ 1 & t_f \end{bmatrix}^{-1} \begin{bmatrix} q_0 \\ q_f \end{bmatrix}$$

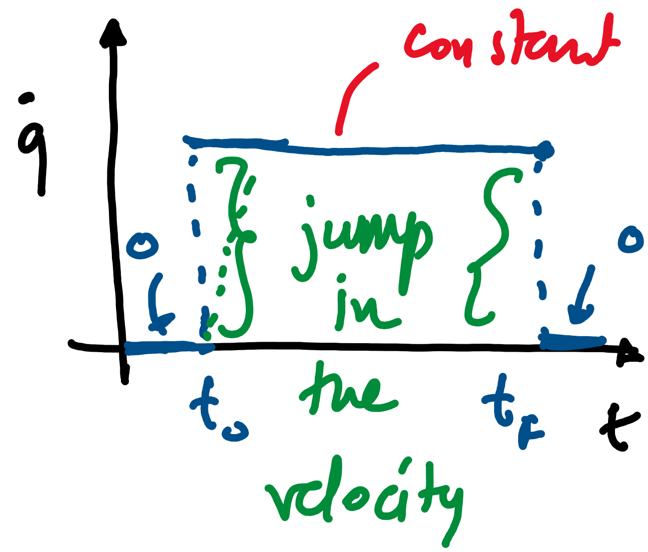
$$= \frac{1}{(t_f - t_0)} \begin{bmatrix} t_f & -t_0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_f \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \left\{ \dfrac{q_0 t_f - q_f t_0}{t_f - t_0} \right\} \\ \left\{ \dfrac{q_f - q_0}{t_f - t_0} \right\} \end{bmatrix}$$

$$\boxed{q(t) = \underbrace{\left(\frac{q_0 t_f - q_f t_0}{t_f - t_0} \right)}_{a_0} + \underbrace{\left(\frac{q_f - q_0}{t_f - t_0} \right)}_{a_1} t}$$

$$\boxed{\underline{\dot{q}} = \left(\frac{q_f - q_0}{t_f - t_0} \right) = \text{constant}}$$





Cubic Profile

To avoid this, we set 4 conditions.

$$q \begin{cases} t = t_0 & q = q_0 \\ t = t_f & q = q_f \end{cases}$$

$$\dot{q} \begin{cases} t = t_0 & \dot{q} = 0 \\ t = t_f & \dot{q} = 0 \end{cases}$$

4 conditions

✓ $q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

✓ $\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$

✓ $\ddot{q}(t) = 2a_2 + 6a_3 t$

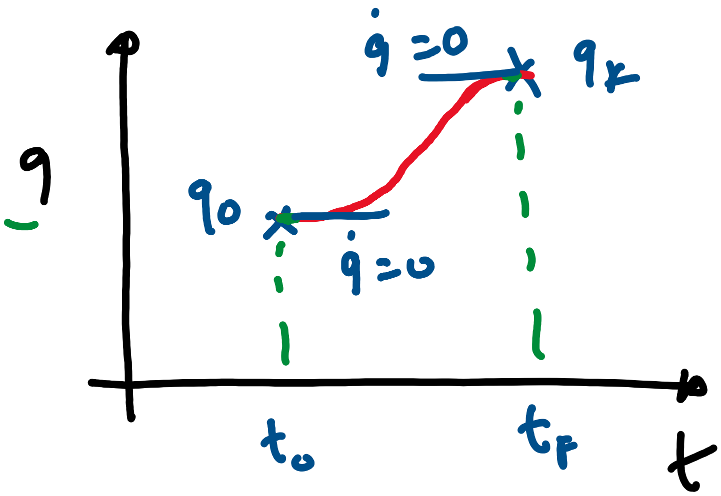
$$\begin{cases} q(t_0) = q_0 \Rightarrow \underline{q_0} = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\ q(t_f) = q_f \Rightarrow \underline{q_f} = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \end{cases}$$

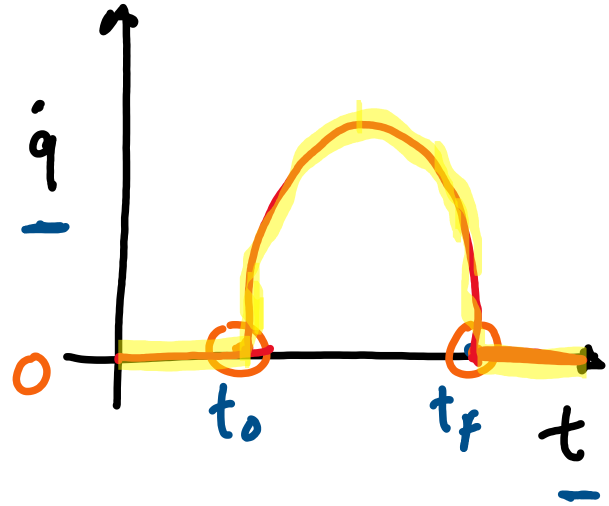
$$\begin{cases} \dot{q}(t_0) = 0 \Rightarrow \underline{0} = a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\ \dot{q}(t_f) = 0 \Rightarrow \underline{0} = a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

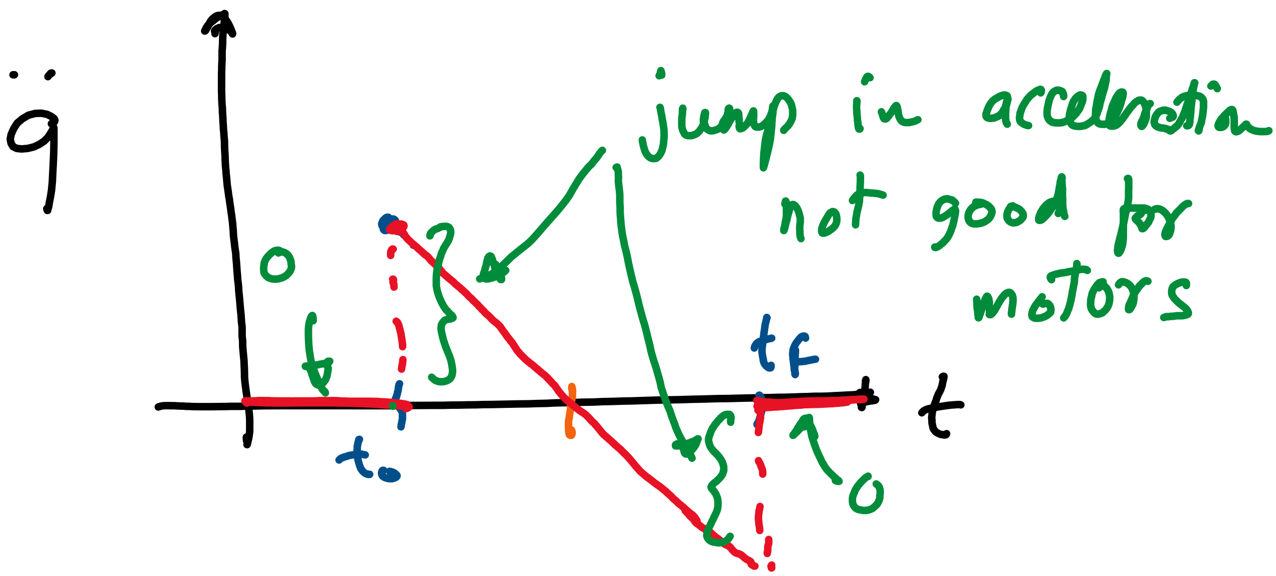
$$\underbrace{\begin{bmatrix} q_0 \\ q_f \\ 0 \\ 0 \end{bmatrix}}_{b\ \checkmark} = \underbrace{\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}}_{A\ \checkmark} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{X\ ?}$$

$$b = AX$$

$$X = A^{-1} b$$


q
q0
q̇=0
q̇=0
qf
t0
tf
t


q̇
0
t0
tf
t


q̈
0
t0
tf
t
0
jump in acceleration
not good for
motors

Quintic Polynomial

To avoid this, we add 2 more conditions

$$\begin{cases} t = t_0 & q = q_0 \\ t = t_f & q = q_f \end{cases} \quad \text{pos.}$$

$$\begin{cases} t = t_0 & \dot{q} = 0 \\ t = t_f & \dot{q} = 0 \end{cases} \quad \text{vel} \qquad \text{6 conditions}$$

$$\begin{cases} t = t_0 & \ddot{q} = 0 \\ t = t_f & \ddot{q} = 0 \end{cases} \quad \text{acc}$$

Assume $q = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$

6 constants

$\dddot{q}$ will be discontinuous.

~ jerk.

add 2 more conditions $\dddot{q}(t_0) = \dddot{q}(t_f) = 0$

this will give a 7th order polynomial

$\ddddot{q}$ — snap

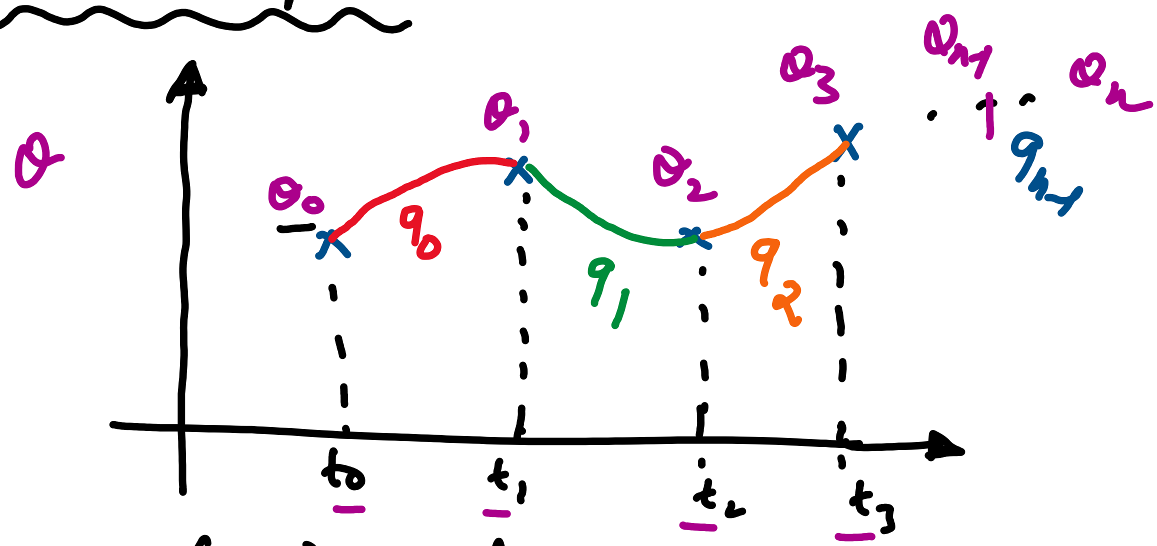
$\overset{\cdot\cdot\cdot\cdot\cdot}{q}$ — crackle

$\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{q}$ — pop

Manipulator — 5^{th} order polynomial
quintic

Drone — 7^{th} order polynomial
septic

Piecewise splines



Given $(n+1)$ data points

$[t_0, \theta_0], [t_1, \theta_1], [t_2, \theta_2], [t_3, \theta_3] \ldots [t_n, \theta_n]$

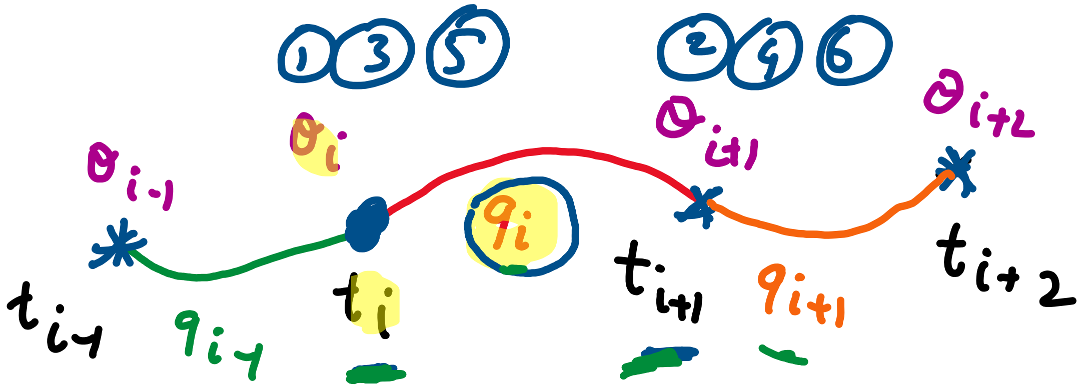
Assume a 3^{rd} order polynomial q_i between $[t_i, \theta_i]$ and $[t_{i+1}, \theta_{i+1}]$

$$q_i = a_{i0} + a_{i1}(t-t_i) + a_{i2}(t-t_i)^2 + a_{i3}(t-t_i)^3$$

We have n third order polynomials,

$q_0, q_1, \ldots q_{n-1}$

$4n$ constants



$$q_i = a_{i0} + a_{i1}(t-t_i) + a_{i2}(t-t_i)^2 + a_{i3}(t-t_i)^3$$

$$q_i(t_i) = a_{i0} = \theta_i \quad (1)$$

$$q_i(t_{i+1}) = a_{i0} + a_{i1}(t_{i+1}-t_i) + \dots \quad (2)$$
$$a_{i2}(t_{i+1}-t_i)^2 + a_{i3}(t_{i+1}-t_i)^3 = \theta_{i+1}$$

$$q_i'(t_i) = q_{i-1}'(t_i) \quad (3)$$

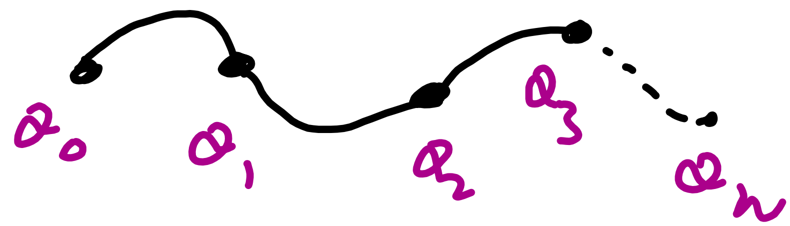
$$\underline{q_i''}(t_i) = \underline{q_{i-1}''}(t_i) \quad (5)$$

$$q_i'(t_{i+1}) = q_{i+1}'(t_{i+1}) \quad (4)$$

$$q_i''(t_{i+1}) = q_{i+1}''(t_{i+1}) \quad (6)$$

Compute the # of constants and # of equations

$n+1$ points



n 3rd order polynomials q_i

4 constant for every 3rd order poly.

$a_{i0}, a_{i1}, a_{i2}, a_{i3}$

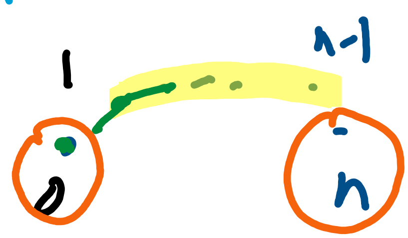
constants : $4n$

$2(n-1)$ position equations (1 to $n-1$)

$n-1$ velocity equations

$n-1$ acceleration equations

2 Q_0 and Q_n



$4n-2$

equations : $4n-2$

4 n constants > 4n−2 equations

We need to set 2 more equations to compute all constants.

Here are few ways of imposing the 2 conditions

① Natural spline

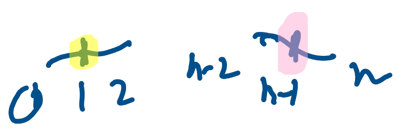
$$q_0''(t_0) = 0 \quad \& \quad q_{n-1}''(t_n) = 0$$

② Clamped condition

$$q_0'(t_0) = 0 \quad \& \quad q_{n-1}'(t_n) = 0$$

③ Not-a-Knot condition

$$q_0'''(t_1) = q_1'''(t_1) \quad \& \quad q_{n-2}'''(t_{n-1}) = q_{n-1}'''(t_{n-1})$$



② Cartesian space (x_c)

(i) position (3D) $[x, y, z]$

position/angle (2D) $[x, y, \theta]$

Same as joint space, use linear, cubic, quintic polynomials, or splines.

Once we get a profile for end-effector, use the inverse kinematics to compute the joint angles

$$q = FK^{-1}(x, y, z) = IK(x, y, z)$$

or

$$\boxed{q = FK^{-1}(x, y, \theta) = IK(x, y, \theta)}$$

code

$$\boxed{q = FK^{-1}(x, y, \theta) = IK(x, y, \theta)}$$

or

code

(ii) Orientation:

a) Euler angles:

$t = t_i \qquad e_i = [\phi_i, \theta_i, \psi_i]$

$t = t_f \qquad e_f = [\phi_f, \theta_f, \psi_f]$

Rescale time $t' = \dfrac{t - t_i}{t_f - t_i}$

$$\boxed{e(t') = (1 - t')e_i + t' e_f}$$

Issues

① Gimbal lock. e.g. $\theta = \pi/2$ (1-2-3)

② Discontinuities due to wrapping of angles at 2π

③ May not give the shortest path.

(b) Rotation matrices

$t_i \; : \; R_i$

$t_f \; : \; R_f$

$$t' = \frac{t - t_i}{t_f - t_i}$$

$$\boxed{R(t') = (1-t') R_i + t' R_f}$$

This does ensure $R^T R = I$

to fix this use SVD

$$U, S, V^T = \text{svd}(R)$$

$$\Rightarrow \; U S V^T = R$$

U, V are orthornormal matrices

S is diagonal matrix, that contains the length parameters.

(i) to make R orthornormal: $\boxed{R = UV^T}$

But $\det(R) = \det(UV^T) = -1$

(ii) To make $\det(R) = 1$ simply flip the sign of the last column of U $\Rightarrow$ U[:,-1] *= -1

(i) and (ii) are done at every interpolation step.

Issues:

① May not give the shortest path

(c) Quaternions

$t_i \; : \; q_i$

$t_f \; : \; q_f$

$$t' = \frac{t - t_i}{t_f - t_i}$$

$$\boxed{\underline{q}(t') = (1-t')\, q_i + t'\, q_f}$$

This does not ensure $|q| = 1$

This is fixed by normalizing at each time step

$$q(t') = \frac{q(t')}{|q(t')|}$$

Issue

(1) Leads to non-constant angular velocity (although angles used linear interpolation)

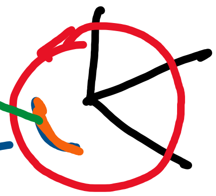
ω, ω_b

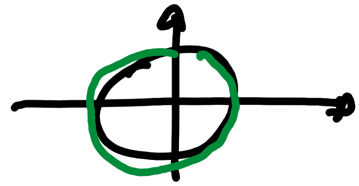
$\omega = 2 \underline{\dot{q}} \cdot \underline{\bar{q}}$, $\omega_b = 2 \bar{q} \cdot \dot{q}$ $\quad \dot{q} =$ constant

$q : [q_0, q_1, q_2, q_3]$ $\qquad$ x, y

$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ $\qquad$ $x^2 + y^2 = 1$

SLERP



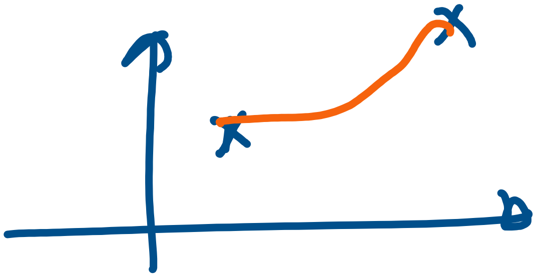


4D sphere

$|q| = 1$

Geodesic path (orientation should follow this path)

$\neq$ Euclidean



Spherical Linear Interpolation (SLERP)

LERP $\quad q = (1-t')q_i + t' q_f$

SLERP $\quad q = \dfrac{\sin(1-t')\theta}{\sin\theta} q_i + \dfrac{\sin t'\theta}{\sin\theta} q_f$

$$\theta = \cos^{-1}(q_i \circ q_f)$$

SLERP fixes both issues of LERP

(i) $|q_{slerp}| = 1$ while $|q_{lerp}| \neq 1$

(ii) ω_{slerp} = constant while

$\omega_{lerp} \neq$ constant.

Note: $\omega = 2\dot{q} \circ \bar{q}$ & $\omega_b = 2\bar{q} \circ \dot{q}$

We can also calculate acceleration

$$\dot{\omega} = 2\ddot{q} \circ \bar{q} + 2|\dot{q}|^2$$

$$\dot{\omega}_b = 2\bar{q} \circ \ddot{q} + 2|\dot{q}|^2$$

$(0, \omega_x, \omega_y, \omega_z)$

$(\underline{0}, w_x, w_y, w_z)$

LERP / SCERP with time scaling

To enforce a velocity / acceleration profile one can scale the time as follows.

Note that $t' = \dfrac{t - t_i}{t_f - t_i}$ is such that

At $t = t_i \quad t' = 0$

$t = t_f \quad t' = 1$

Lets choose $s(t')$ such that

$$\left.\begin{aligned} &s(t'=0)=0, \quad s(t')=1 \\ &\dot{s}(t'=0)=\dot{s}(t'=1)=0 \\ &\ddot{s}(t'=0)=\ddot{s}(t'=1)=0 \end{aligned}\right\} \text{6 conditions}$$

$$s(t) = \underbrace{a_0 + a_1(t') + a_2(t')^2 + a_3(t')^3 + a_4(t')^4 + a_5}_{\text{6 constants}}(t')^5$$

Solve for the 6 constants using the 6 conditions gives

$$s(t') = 6(t')^5 - 15(t')^4 + 10(t')^3$$

Now use LERP / SLERP

$$\boxed{\begin{aligned} q_{lerp}(t') &= (1-s(t'))\,q_i + s(t')\,q_f \\ q_{slerp}(t') &= \frac{\sin(1-s(t'))\theta}{\sin\theta}\,q_i + \frac{\sin(s(t'))\theta}{\sin\theta}\,q_f \end{aligned}}$$