Quaternions

4D vector: $q = q_0 + q_x \hat{i} + q_y \hat{j} + q_z \hat{k}$

Various ways of writing q.

① $q = (q_0, \vec{q})$

② $Re(q) = q_0$

$Im(q) = \vec{q} = (q_x, q_y, q_z)^T$

③ $q = \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{bmatrix}$

Conjugate of q is $\bar{q} = (q_0, -\vec{q})$

Norm $|q| = \sqrt{q_0^2 + q_x^2 + q_y^2 + q_z^2}$

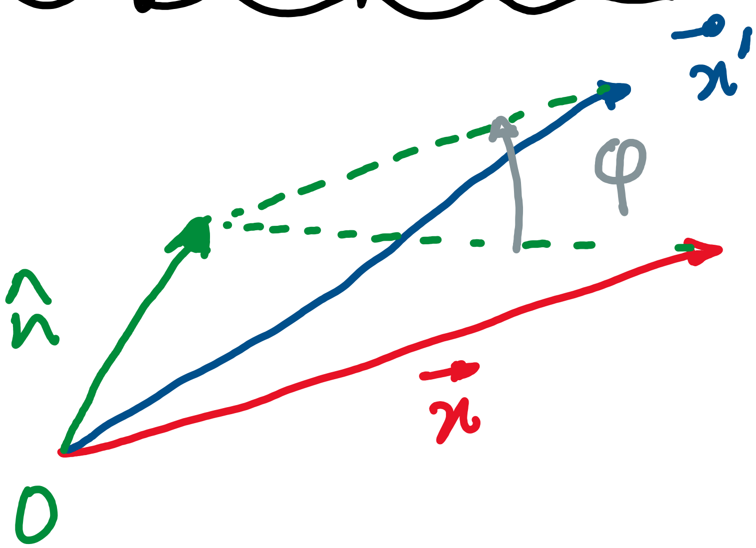
Quaternion Product

$$\boxed{q \cdot p = (q_0, \vec{q}) \cdot (p_0, \vec{p}) = (q_0 p_0 - \vec{p} \cdot \vec{q},\ q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p})}$$

dot product　　vector product

dot product   vector product

Axis - Angle Representation



Vector $\vec{x}$ rotated to $\vec{x}'$

This rotation may be expressed by a unit vector $\hat{n}$ passing through the origin O and an angle φ as shown in the figure

$\hat{n} - \varphi$ is the axis-angle representation for rotation

Axis-angle $(\hat{n}-\varphi) \Longrightarrow R$

$$\vec{x} = R\,\vec{x}\,'$$

$$\boxed{R = I + \sin\varphi\, N + (1-\cos\varphi)\, N^2}$$

$$N = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} \qquad \hat{n} = (n_x, n_y, n_z)$$

$$N^2 = (N)(N)$$

$$\boxed{R^T = I - \sin\varphi\, N + (1-\cos\varphi)\, N^2}$$

$$\vec{x}\,' = R^T \vec{x}$$

Rodriguez Rotation Formula

$$\hat{n}, \varphi \Longrightarrow R$$

$$R \not\Longrightarrow \hat{n}, \varphi \qquad \text{complex}$$

quat

Quaternion $(q) \Longleftrightarrow$ Axis-angle $(\hat{n}\ \varphi)$

$$(\underline{q_0}, \underline{\underline{\vec{q}}}) = \left(\underline{\cos}\left(\frac{\varphi}{2}\right), \sin\left(\frac{\varphi}{2}\right)\hat{n}\right)$$

Given $q = (q_0, \vec{q})$ one can compute $\hat{n}, \varphi$ as follows

$$\boxed{\begin{aligned} \varphi &= 2\cos^{-1}(q_0) \\ \hat{n} &= \left[\vec{q} \,/\, \sin(\varphi/2)\right] \end{aligned}}$$

Given $\hat{n}, \varphi$ one can compute q_0, q as follows

$$\boxed{\begin{aligned} q_0 &= \cos(\varphi/2) \\ \vec{q} &= \sin(\varphi/2)\,\hat{n} \end{aligned}}$$

Easy to compute either way $q_0, \vec{q} \Longleftrightarrow \hat{n}, \varphi$

Quaternion $\Rightarrow$ Rotation

It can be shown that

$$x = q \cdot x' \cdot \bar{q} \iff x = R x'$$

where $x = (0, \vec{x})$ $\quad x' = (0, \vec{x}')$

$$\boxed{R = \begin{bmatrix} q_0^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_0 q_z) & 2(q_x q_z + q_0 q_y) \\ 2(q_x q_y + q_0 q_z) & q_0^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_z q_y - q_0 q_x) \\ 2(q_x q_z - q_0 q_y) & 2(q_z q_y + q_0 q_x) & q_0^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}}$$

NOTE: Given q_0, q_x, q_y, q_z it is straightforward to compute R

Rotation $\Rightarrow$ Quaternion

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Step 1 : Compute magnitude of each component of the quaternion

$$|q_0| = \sqrt{\frac{1 + r_{11} + r_{22} + r_{33}}{4}}$$

$$|q_x| = \sqrt{\frac{1 + r_{11} - r_{22} - r_{33}}{4}}$$

$$|q_y| = \sqrt{\frac{1 - r_{11} + r_{22} - r_{33}}{4}}$$

$$|q_z| = \sqrt{\frac{1 - r_{11} - r_{22} + r_{33}}{4}}$$

Step 2: Find the largest component

1. If q_0 is largest

$$q_x = \frac{r_{32} - r_{23}}{4q_0}, \quad q_y = \frac{r_{13} - r_{31}}{4q_0}, \quad q_z = \frac{r_{21} - r_{12}}{4q_0}$$

2. If q_x is largest

$$q_0 = \frac{r_{32} - r_{23}}{4q_x}; \quad q_y = \frac{r_{12} + r_{21}}{4q_x}; \quad q_z = \frac{r_{13} + r_{31}}{4q_x}$$

3. If q_y is largest

$$q_0 = \frac{r_{13} - r_{31}}{4q_y}; \quad q_x = \frac{r_{12} + r_{21}}{4q_y}; \quad q_z = \frac{r_{23} + r_{32}}{4q_y}$$

4. If q_z is largest

$$q_0 = \frac{r_{21} - r_{12}}{4q_z}; \quad q_x = \frac{r_{13} + r_{31}}{4q_z}, \quad q_y = \frac{r_{23} + r_{32}}{4q_z}$$

The reason is because (q_0, q_x, q_y, q_z) & $(-q_0, -q_x, -q_y, -q_z)$ denote the same rotations.

Euler angles $\Rightarrow$ Quaternion

If ϕ, θ, ψ are 1-2-3 Euler angles then we wrote the net rotation as

$$R = R_x(\phi)\, R_y(\theta)\, R_z(\psi)$$

If ϕ, θ, ψ are 1-2-3 Euler angles then we can write net quaternion as follows

quaternion dot product

$$q = q_1 \bullet q_2 \bullet q_3$$

where $q_1 = [\cos(\phi/2), \sin(\phi/2)\hat{i}]$

$$q_2 = [\cos(\theta/2), \sin(\theta/2)\hat{j}]$$

$$q_3 = [\cos(\psi/2), \sin(\psi/2)\hat{k}]$$

$\hat{i} = [1,0,0)$ $\hat{j} = [0,1,0]$ $\hat{k} = [0,0,1]$

Summary

We have 4 ways of representing rotations. R, q, Euler, $\hat{n}-\varphi$

1. Given q, Euler, $\hat{n}-\varphi$ it is easy to compute R but not vice versa
2. R– Euler is the easiest (HW question)
3. R- quaternion is complex but doable
4. q to $\hat{n}-\varphi$ is easy both ways.