Euler Angles

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler angles are used to describe arbitrary orientation of an object

1-2-3 Bryant angles (MuJoCo)

x-y-z	y-x-z	z-y-x
x-z-y	y-z-x	z-x-y
x-y-x	y-z-y	z-y-z
x-z-x	y-x-y	z-x-z

Tait-Bryant angles aerospace (3-2-1)

12 unique ways of describing rotations.

1-2-3 Euler angles

$$R = R_x(\phi)\, R_y(\theta)\, R_z(\psi)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c\phi = \cos\phi$ and so on

$s\phi = \sin\phi$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$r_{11} = c\psi\, c\theta$; $r_{21} = s\phi\, s\theta\, c\psi + s\psi\, c\phi$

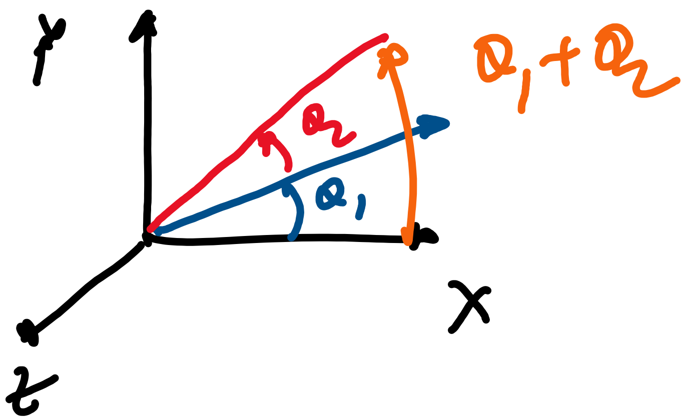
$r_{31} = c\phi\, s\theta\, c\psi - s\psi\, s\phi$

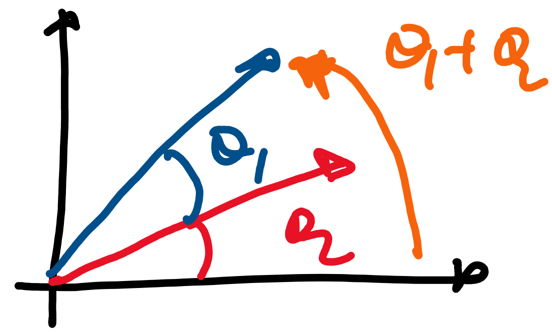
$r_{12} = -s\psi\, c\theta$; $r_{22} = -s\phi\, s\psi\, s\theta + c\phi\, c\psi$

$r_{32} = s\phi\, c\psi + s\psi\, s\theta\, c\phi$

$r_{13} = s\theta$; $r_{23} = -s\phi\, c\theta$; $r_{33} = c\phi\, c\theta$

2D rotations commute



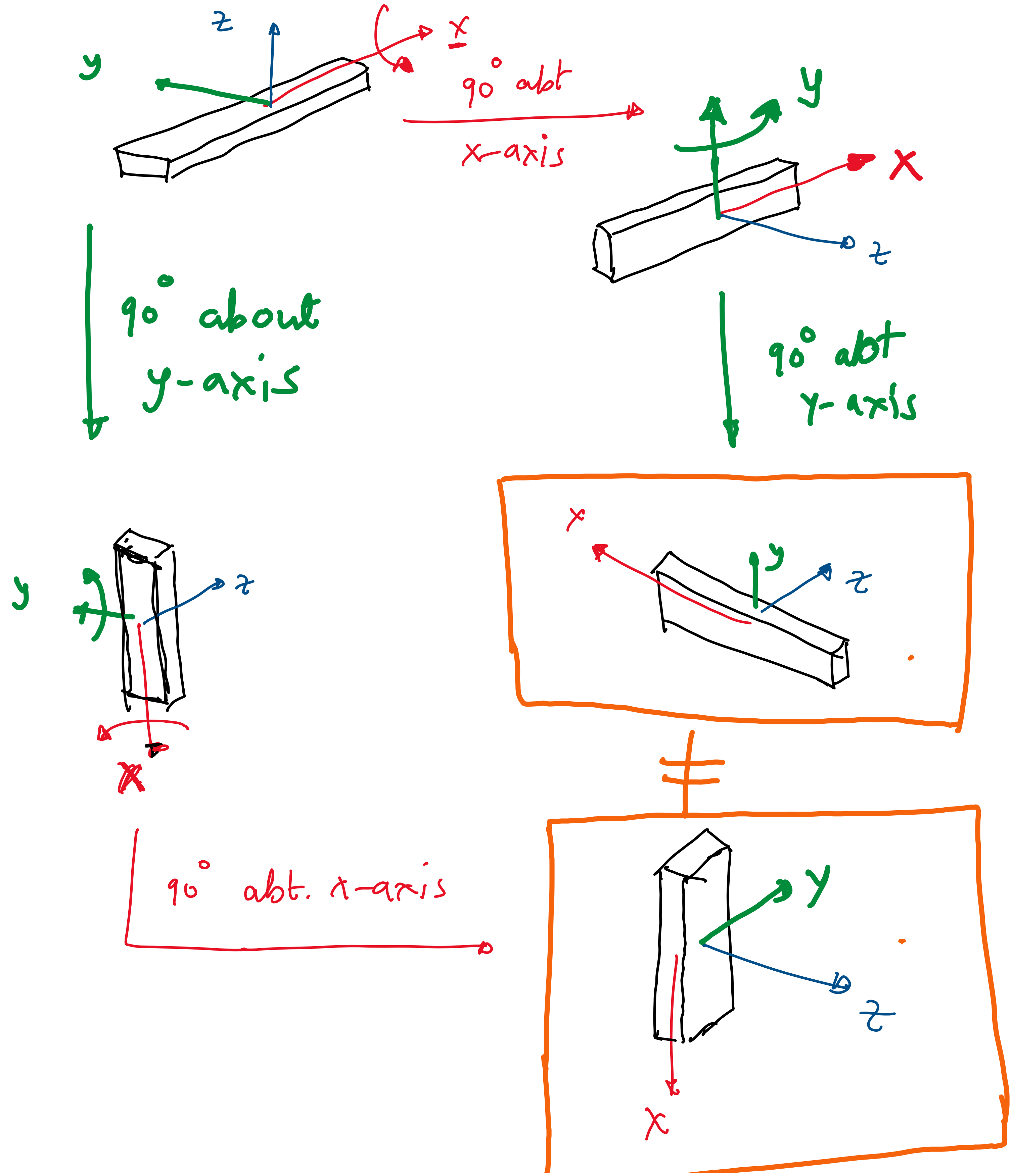


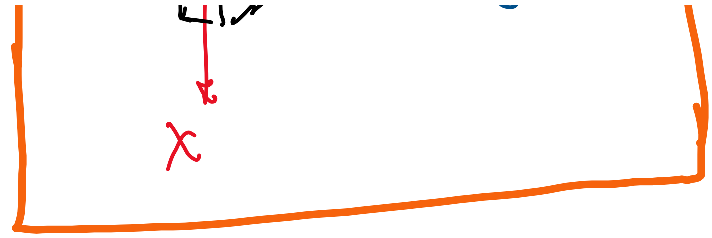
$$R_z(\theta_1)\,R_z(\theta_2) = R_z(\theta_1+\theta_2)$$

$$R_z(\theta_2)\,R_z(\theta_1) = R_z(\theta_1+\theta_2)$$

Rotations in 2D commute

3D rotations do not commute




x

Gimball Lock

Gimball lock is the loss of a degree of freedom. This happens when 2 axis are parallel to each other.

This happens due to the use of Euler angles.

$$R = R_x(\phi)\, R_y(\pi/2)\, R_z(\psi)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ -\cos(\phi+\psi) & \sin(\phi+\psi) & 0 \end{bmatrix}$$

① $\phi = 0 \quad \psi = \pi/2 \quad R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

② $\phi = \pi/2 \quad \psi = 0 \quad R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

③ $\phi = \psi = \pi/4 \quad R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Since these give the same R matrix we are unable to distinguish these 3 different rotations from each other

This is the gimbal lock.

This happen because of the use of Euler angles.

This can be fixed using Quaternions