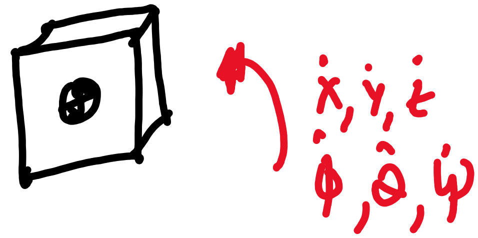
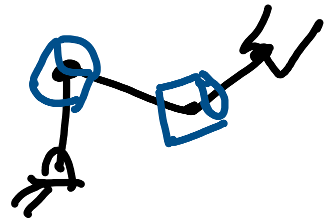
Differential Kinematics

① Free joint



② Revolute / Prismatic joint



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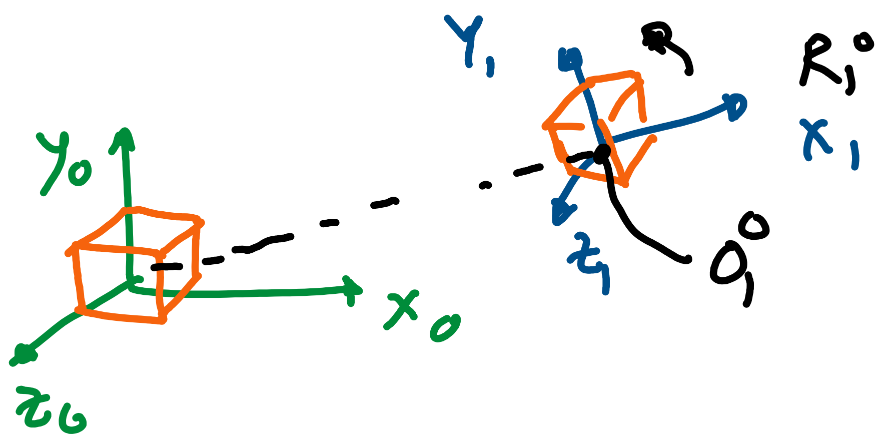
① Free Joint

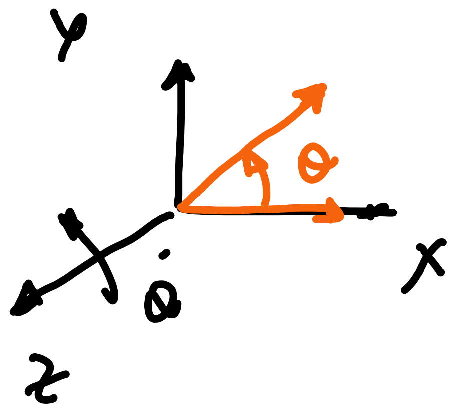


$$H_1^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}$$

Linear velocity $\dot{O}_1^0 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$

Angular velocity $\dot{R}_1^0 \Rightarrow$ How is this related to ω

2D vs 3D

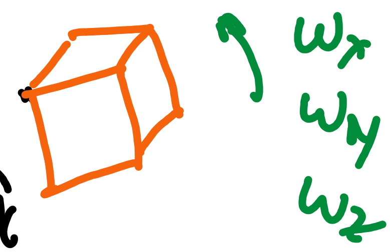


In 2D

$$\vec{\omega}_z = \dot{\theta}\,\hat{k}$$

unit vector along z

In 3D

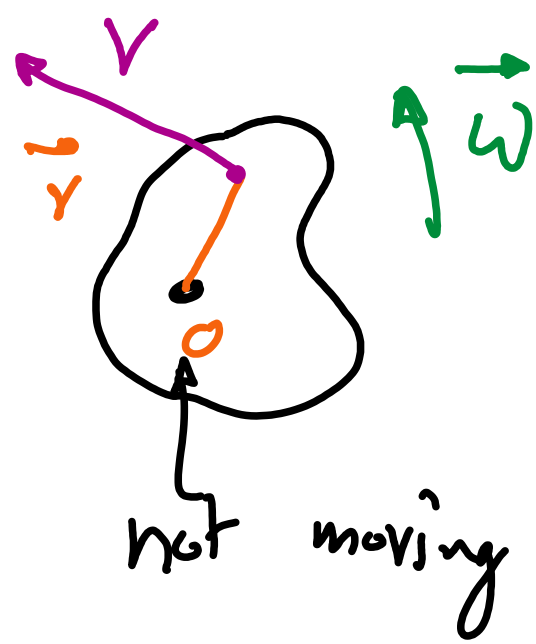


$$\vec{\omega} \neq \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

does not hold true

In 2D & 3D

$$\vec{v} = \underline{\vec{\omega}} \times \underline{\vec{r}}$$



Relate $\dot{R}$ to ω

$$RR^T = I$$

Derivative wrt time

$$\dot{R}R^T + R\dot{R}^T = 0$$

Use the property $(AB)^T = B^T A^T$

$$A = (A^T)^T$$

$$\dot{R}R^T + \left(\left(\underbrace{R\dot{R}^T}\right)^T\right)^T = 0$$

$$\underbrace{\dot{R}R^T} + \left(\underbrace{\dot{R}R^T}_{S}\right)^T = 0$$

$$\boxed{\dot{R}R^T = S(a)}$$

$$\underline{S}(a) + \underline{S}^T(a) = 0$$

S = skew symmetric matrix

e.g. $\underset{[a_x,\, a_y,\, a_z)}{S(a)} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$

$$\dot{R} R^T = S(a)$$

Post multiply by R

$$\dot{R} \underbrace{R^T R}_{I} = S(a) R$$

$$\boxed{\dot{R} = S(a) R} \quad \text{— (I)}$$

What is $a = (a_x, a_y, a_z)$

$$r = \underline{R\, r^b} \quad \text{— (II)}$$

r - position in world frame

r_b - position is the body frame

Differentiate with respect to time

$$\dot{r} = \dot{R} r^b + R \cancelto{0}{\dot{r}^b}$$

From (II)

From (I) $\dot{r} = S(a) \underline{R\, r^b} = S(a) R$

$$\dot{r} = S(a)\, r$$

But $\underbrace{a \times b}_{3\times1} = \underset{3\times3}{S(a)}\ \underset{3\times1}{b}$

$$\dot{\vec{r}} = \vec{a} \times \vec{r}$$

But we know that $\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$

Thus $\vec{a} = \vec{\omega}$

$$\boxed{\begin{aligned} \dot{R} &= S(a)R = S(\omega)R \\ S(\omega) &= \begin{bmatrix} 0 & -\omega_x & \omega_y \\ \omega_x & 0 & -\omega_z \\ -\omega_y & \omega_z & 0 \end{bmatrix} \end{aligned}}$$

Relate the angular velocity $\omega_x, \omega_y, \omega_z$
to rate of change of Euler angle $\dot{\phi}, \dot{\theta}, \dot{\psi}$

$$\dot{R} = S(\omega) R \Rightarrow S(\omega) = \dot{R} R^T \quad — (*)$$

Bryant 1-2-3

Let $R = R_x R_y R_z$; $R^T = R_z^T R_y^T R_x^T$

$$\dot{R} = \dot{R}_x R_y R_z + R_x \dot{R}_y R_z + R_x R_y \dot{R}_z$$

$$(*) \; S(\omega) = \underbrace{(\dot{R}_x R_y R_z) R_z^T R_y^T R_x^T}_{1} +$$

$$\underbrace{(R_x \dot{R}_y R_z)(R_z^T R_y^T R_x^T)}_{2} +$$

$$(R_x R_y \dot{R}_z)(R_z^T R_y^T R_x^T) \; 3$$

1 $\dot{R}_x \underbrace{R_y \underbrace{R_z R_z^T}_{= I} R_y^T}_{= I} R_x^T = \dot{R}_x R_x^T = S(\dot{\phi} \hat{i})$

2 $R_x \dot{R}_y \underbrace{R_z R_z^T}_{= I} R_y^T R_x^T = R_x \underbrace{\dot{R}_y R_y^T}_{S(\dot{\theta}\hat{j})} R_x^T$

$= R_x S(\dot{\theta} \hat{j}) R_x^T = S(R_x \dot{\theta} \hat{j})$

$\longrightarrow R \, S(a) R = S(Ra)$

$RS(a)R = S(Ra)$

③ $R_x R_y \underbrace{\dot{R}_z R_z^T}_{S(\dot{\psi}\hat{k})} R_y^T R_x^T = R_x R_y S(\dot{\psi}\hat{k}) R_y^T R_x^T$

$$= (R_x R_y)\, S(\dot{\psi}\hat{k})\, (R_x R_y)^T \qquad (AB)^T = B^T A^T$$

$$= S(R_x R_y \dot{\psi}\hat{k}) \qquad \text{As } R\, S(a) R^T = S(Ra)$$

From ①, ②, ③

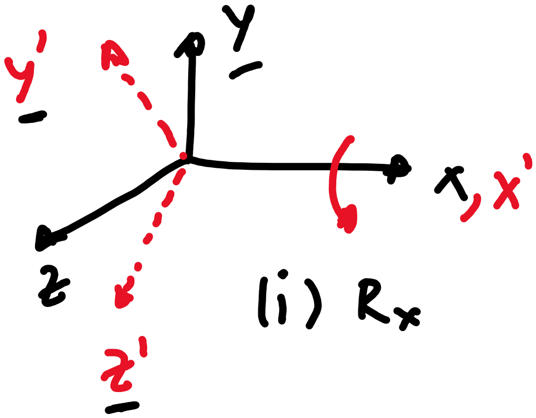
$$S(\omega) = S(\dot{\phi}\hat{i}) + S(R_x \dot{\theta}\hat{j}) + S(R_x R_y \dot{\psi}\hat{k})$$

$$S(\omega) = S(\dot{\phi}\hat{i} + R_x \dot{\theta}\hat{j} + R_x R_y \dot{\psi}\hat{k})$$

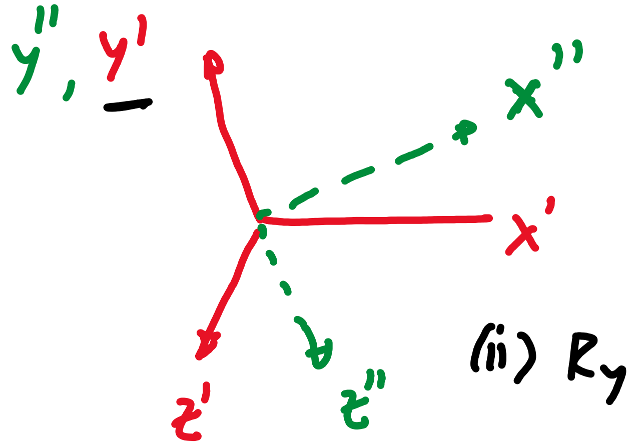
$$\boxed{\vec{\omega} = \dot{\phi}\hat{i} + R_x \dot{\theta}\hat{j} + R_x R_y \dot{\psi}\hat{k}}$$

This is true for 1-2-3 Euler Angles.

Another method to relate $\vec{\omega}$ with $\dot{\phi}, \dot{\theta}, \dot{\psi}$



(i) R_x



(ii) R_y

$$\vec{\omega} = \dot{\phi}\,\underline{x} + \dot{\theta}\,\underline{y}' + \dot{\psi}\,\underline{z}''$$

} see angular velocity in diff frames.

$$\boxed{\omega_3^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2}$$

$$y' = \underline{R_x}\, y \qquad \text{from (i)}$$

$$z'' = R_x (R_y z)$$

$$\omega_3^0 = \dot{\phi} x + R_x \dot{\theta} y + R_x R_y \dot{\psi} z$$

$$\vec{\omega} = \dot{\phi}\, x + R_x \dot{\theta}\, y + R_x R_y \dot{\psi}\, z$$

$$= \dot{\phi}\,\hat{i} + R_x \dot{\theta}\,\hat{j} + R_x R_y \dot{\psi}\,\hat{k}$$

$$= \dot{\phi}\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & & \\ 0 & & \end{bmatrix}\dot{\theta}\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\dot{\psi}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{\omega} = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\vec{\omega} = \underline{A}\,\dot{\bar{\Theta}}$$

$$\det A = \cos\theta$$

$$\dot{\Theta} = A^{-1}\omega$$

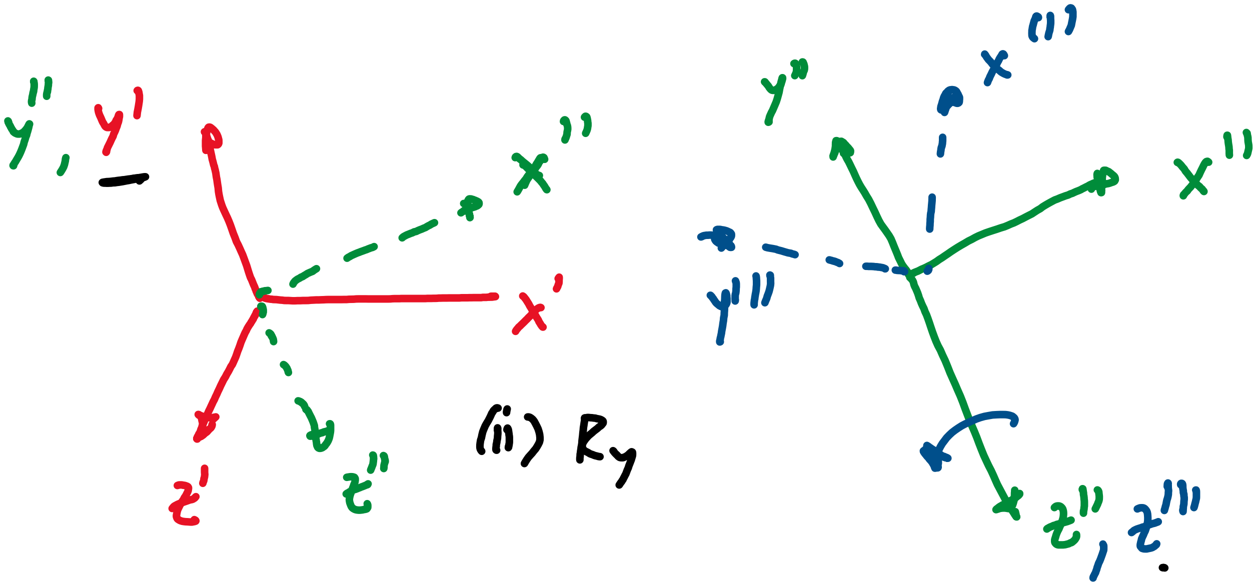
$\rightarrow$ Not defined when

$\det(A) = \cos\theta = 0$

$\theta = \pi/2$

Singularity / Gimbal lock.

(ii) R_y



$$\Omega = \dot{\psi}\, z''' + \dot{\theta}\, y' + \dot{\phi}\, x$$

$$= \dot{\psi}\, z''' + \dot{\theta}\, R_z^T\, y''' + \dot{\phi}\, R_z^T R_y^T\, x'''$$

$$= \dot{\psi}\begin{pmatrix}0\\0\\1\end{pmatrix} + \dot{\theta}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} + \dot{\phi}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\boxed{\begin{aligned}\Omega &= \begin{bmatrix}\cos\psi\cos\theta & \sin\psi & 0\\ -\sin\psi\cos\theta & \cos\psi & 0\\ \sin\theta & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}\\ \Omega &= B\,\dot{\Theta}\end{aligned}}$$

$$\det B = \cos\theta$$

The inverse does not exist when $\theta = 90^\circ$.

Quaternions related to Angular Velocity

$$x = q \circ x' \circ \bar{q}$$

$$x' = \bar{q} \circ x \circ q$$

x, x' position in world and body frame respectively.

$x = [0, \vec{x}]$; $x' = [0, \vec{x}']$

It can be shown that

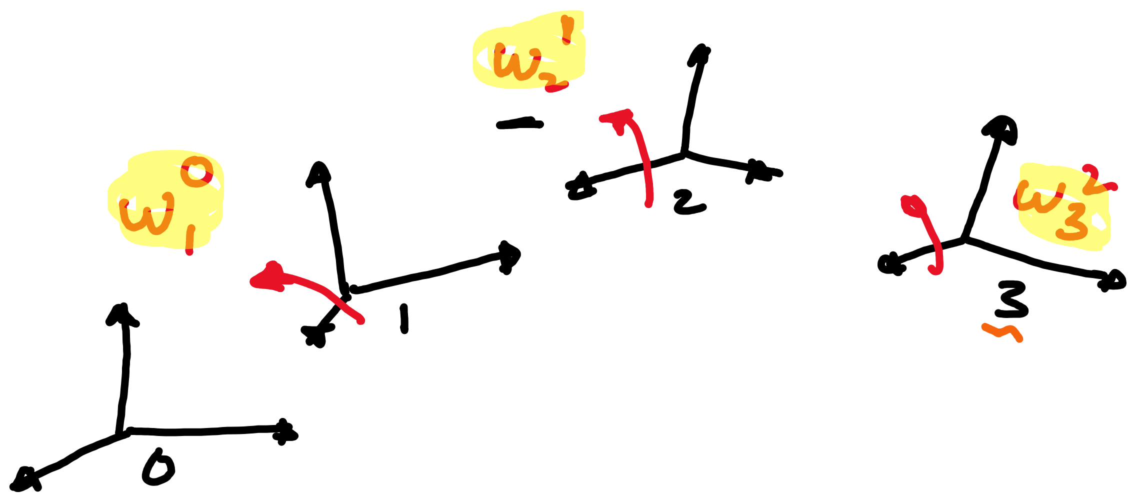
$$\boxed{\begin{aligned} \dot{q} &= \frac{1}{2} \omega \circ q \\ \dot{q} &= \frac{1}{2} q \circ \Omega \end{aligned}}$$

ω - angular velocity in world frame

Ω - angular velocity in body frame

$\omega = (0, \vec{\omega})$, $\Omega = [0, \vec{\Omega}]$

Angular velocity in different frames



frame 0

$$\left.\begin{cases} \omega_3^0 = \underbrace{\omega_1^0 + R_1^0 \omega_2^1}_{} + R_2^0 \omega_3^2 \\ \omega_2^0 = \underbrace{\omega_1^0 + R_1^0 \omega_2^1}_{} \end{cases}\right\} \circledast$$

From $\circledast$

$$\omega_3^0 = \omega_2^0 + \underbrace{R_2^0 \omega_3^2}_{}$$

Generalization of angular velocity

$$R_n^0 = R_1^0 R_2^1 R_3^2 + \dots R_n^{n-1}$$

$$\boxed{\omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2 + \dots + R_{n-1}^0 \omega_n^{n-1}}$$

Here ω_i^{i-1} is the relative angular velocity of frame i with respect to frame $i-1$ (previous frame)

The above formula can also be applied recursively as follows

$$\boxed{\begin{aligned} \omega_2^0 &= \omega_1^0 + R_1^0 \omega_2^1 \\ \omega_3^0 &= \omega_2^0 + R_2^0 \omega_3^2 \\ \omega_4^0 &= \omega_3^0 + R_3^0 \omega_4^3 \\ &\vdots \\ \omega_n^0 &= \omega_{n-1}^0 + R_{n-1}^0 \omega_n^{n-1} \end{aligned}}$$

$$\boxed{\Omega_n = \omega_n^{n-1} + (R_n^{n-1})^T \omega_{n-1}^{n-2} + (R_{n-1}^{n-2})^T \omega_{n-2}^{n-3} + \ldots + (R_n^1)^T \omega_1^0}$$

Here $R_n^k = R_{k+1}^k R_{k+2}^{k+1} \ldots R_n^{n-1}$

The above formula can also be applied recursively as follows.

$$\boxed{\begin{aligned}
\Omega_1 &= \omega_1^0 \\
\Omega_2 &= \omega_2^1 + (R_2^1)^T \Omega_1 \\
\Omega_3 &= \omega_3^2 + (R_3^2)^T \Omega_2 \\
\Omega_4 &= \omega_4^3 + (R_4^3)^T \Omega_3 \\
&\vdots \\
\Omega_n &= \omega_n^{n-1} + (R_n^{n-1})^T \Omega_{n-1}
\end{aligned}}$$