Jacobian (Gradient)

$$f(q) = [f_1(q) \quad f_2(q) \quad \ldots \quad f_m(q)]$$

$$q = [q_1, q_2, \ldots \quad q_n]$$

$$\underset{m\times n}{J} = \frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix}$$

$m \times n$

EXAMPLE: $f = [\underbrace{x^2+y^2}_{f_1}, \underbrace{2x+3y+5}_{f_2}, \underbrace{xy}_{f_3}]$

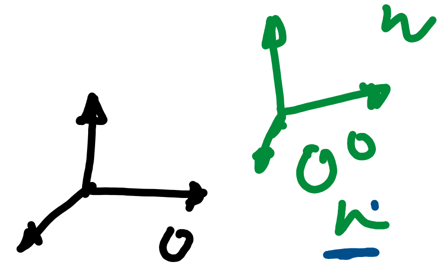
$q = [x, y]$

$$J = \frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2 & 3 \\ y & x \end{bmatrix}$$

$$J\Big|_{\substack{x=1 \\ y=2}} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

① Jacobian to compute the linear/angular velocity of frames/bodies

$$H_n^0 = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$



$$v_n^0 = \dot{o}_n^0 \longrightarrow v_n^0 = J_v \, \dot{q}$$

$$S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T \longrightarrow \omega_n^0 = J_\omega \, \dot{q}$$

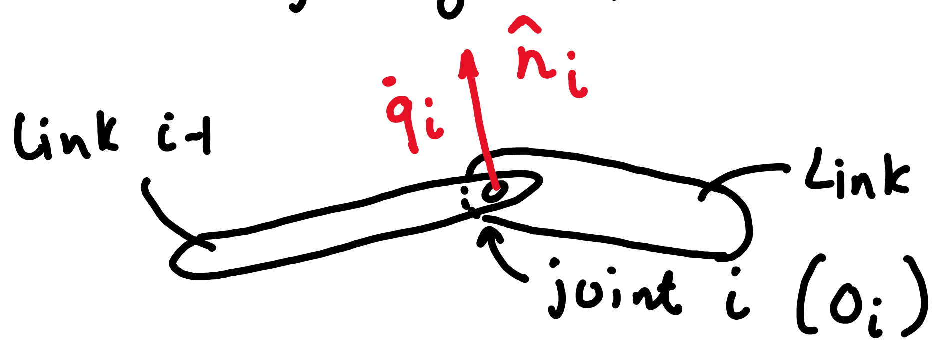
$$\underbrace{\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix}}_{6\times 1} = \underbrace{\begin{bmatrix} J_v \\ J_\omega \end{bmatrix}}_{6\times N} \underbrace{\dot{q}}_{N\times 1}$$

$\dot{q}$ — rate of change of angle/length of a body/chain

How to compute J_v, J_ω numerically/symbolically.

Computing J_ω

Consider 2 adjoining links



The angular velocity of link i wrt. link i-1

(i) joint i is revolute : $\omega_i^{i-1} = \dot{q}_i \hat{n}_i$

(ii) joint i is prismatic : $\omega_i^{i-1} = 0$

We desired an expression for ω_n^0

$$\omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2 + \dots + R_{n-1}^0 \omega_n^{n-1}$$

$$\omega_n^0 = \rho_1 \dot{q}_1 \hat{n}_1 + \rho_2 R_1^0 \dot{q}_2 \hat{n}_2 + \rho_3 R_2^0 \dot{q}_3 \hat{n}_3 + \dots + \rho_n R_{n-1}^0 \dot{q}_n \hat{n}_n$$

$$\omega_n^0 = \underbrace{\begin{bmatrix} \rho_1 \hat{n}_1 & \rho_2 R_1^0 \hat{n}_2 & \rho_3 R_2^0 \hat{n}_3 & \dots & \rho_n R_{n-1}^0 \hat{n}_n \end{bmatrix}}_{J_\omega} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$\rho_i = 0$ prismatic

$\rho_i = 1$ revolute

I

$\rho_i = 0, 1$

prism, revolute

Computing J_v

$$V_n = \dot{o}_n = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i \qquad \text{Chain rule}$$

$$V_n = \underbrace{\begin{bmatrix} \frac{\partial o_n^0}{\partial q_1} & \frac{\partial o_n^0}{\partial q_2} & \cdots & \frac{\partial o_n^0}{\partial q_n} \end{bmatrix}}_{J_v} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

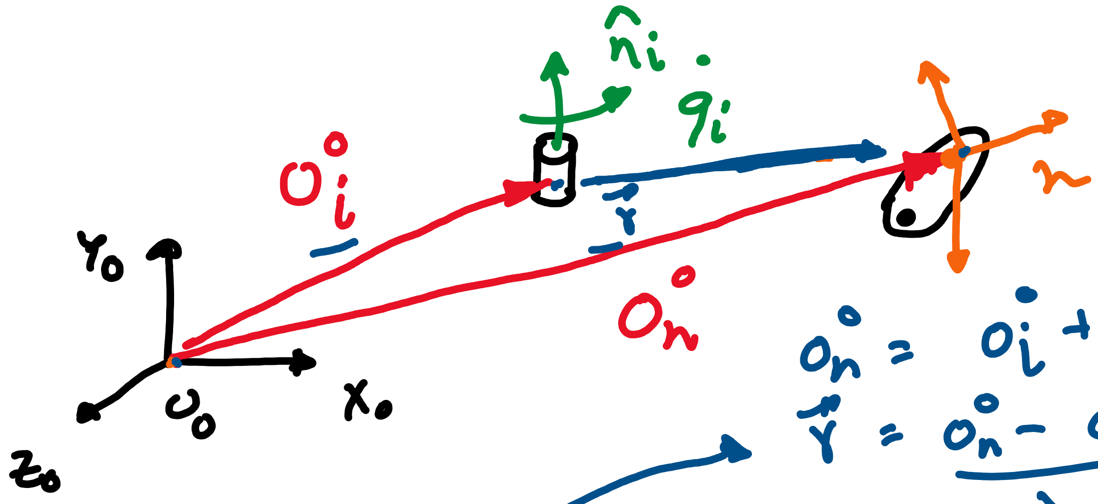
$$\boxed{J_{v_i} = \frac{\partial o_n^0}{\partial q_i}}$$

From the above formula we can see that J_{v_i} may be obtained as $\dot{O}_n = J_{v_i}$ by setting $\dot{q}_i = 1$ and $\dot{q}_j = 0$ $(j \neq i)$

$J_{v_1} = ?$ $\dot{q}_1 = 1$ $\dot{q}_2, \dot{q}_3 = \dots \dot{q}_n = 0$ $J_{v_1} = V_n$

calc?

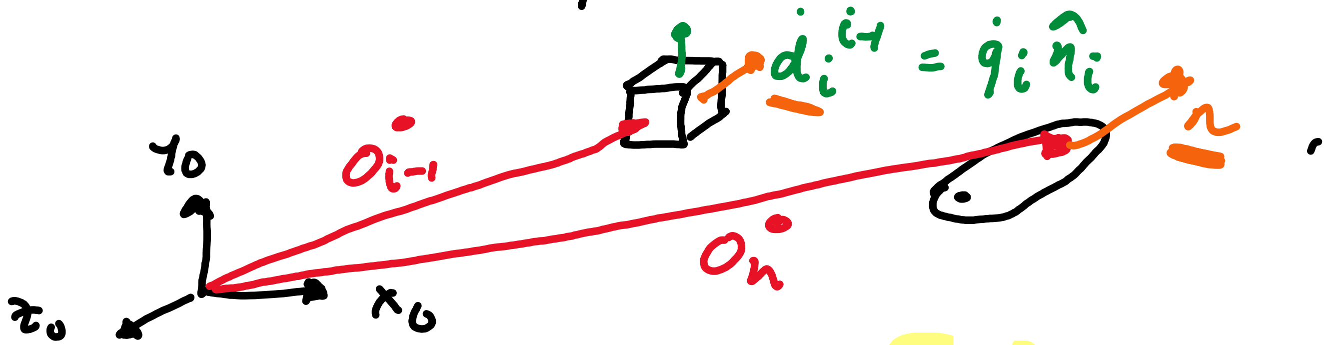
(i) joint i is revolute



$$O_n^0 = O_i^0 + \vec{r}$$
$$\vec{r} = O_n^0 - O_i^0$$

$$\underbrace{\dot{O}_n^0}_{V_n^0} = \vec{\omega} \times \vec{r} = R_{i-1}^0 \, \omega_i^{i-1} \times (O_n^0 - O_i^0)$$
$$= R_{i-1}^0 \, \dot{q}_i \, \hat{n}_i \times (O_n^0 - O_i^0)$$
$$= \underbrace{R_{i-1}^0 \, \hat{n}_i \times (O_n^0 - O_i^0)}_{J_{v_i}} \, \dot{q}_i \quad \text{— II}$$

(ii) joint i is prismatic



$$\dot{O}_n = \dot{d}_i^{i-1} = R_{i-1}^0 \, \dot{q}_i \, \hat{n}_i = \underbrace{R_{i-1}^0 \, \hat{n}_i}_{J_{v_i}} \, \dot{q}_i \quad \text{— III}$$

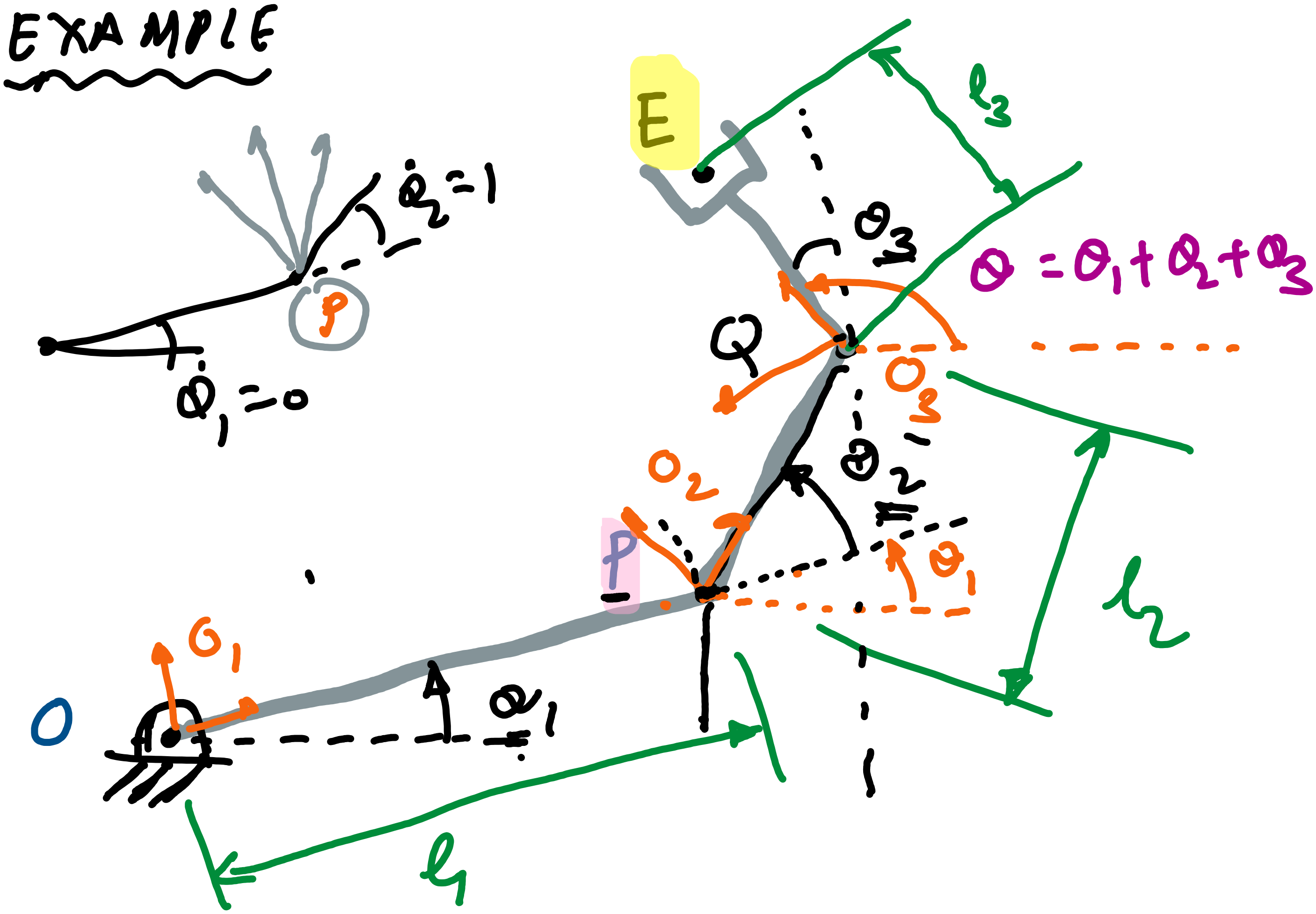
Jacobian Summary

$$J_{v_i} = \begin{cases} R^0_{i-1}\,\hat{n}_i \times (O^0_n - O^0_i) & \text{Revolute} \quad \text{II} \\ R^0_{i-1}\,\hat{n}_i & \text{Prismatic} \quad \text{III} \end{cases}$$

$$J_{\omega_i} = \begin{cases} R^0_{i-1}\,\hat{n}_i & \text{Revolute} \\ 0 & \text{Prismatic} \end{cases} \quad \text{I}$$

$g_i = 0, 1$

EXAMPLE



a) Compute the Jacobian of point E

b) Compute the Jacobian of point P

(a) Jacobian for E

$$\boxed{\begin{aligned} J_{\underline{v_i}} &= \underline{R_{i-1}^0 \hat{n}_i \times (O_{\underline{n}}^0 - O_i^0)} && \text{Revolute} \\ J_{\omega_i} &= \underline{R_{i-1}^0 \hat{n}_i} && \text{Revolute} \end{aligned}}$$

$$\underset{6\times3}{J^E} = \begin{bmatrix} J_v^E \\ J_\omega^E \end{bmatrix}_{6\times3}$$

$$n_i = \hat{k} = [0,0,1]$$

$$J_v^E = [R_0^0 k \times (e^0 - O_0^0), R_1^0 k \times (e^0 - O_1^0), R_2^0 k (e^0 - O_2^0)]$$

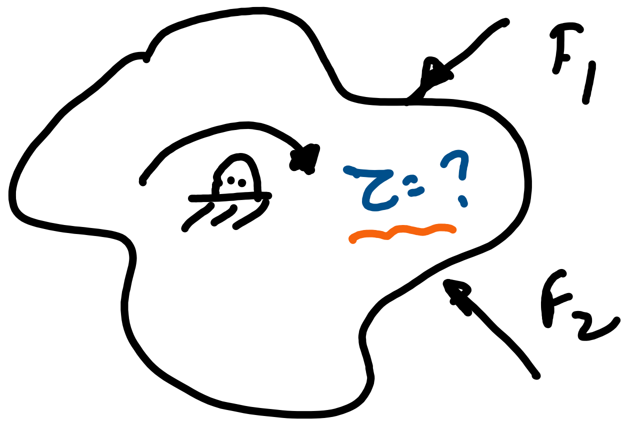
$$J_\omega^E = [R_0^0 k, R_1^0 \hat{k}, R_2^0 \hat{k}]$$

(b) $$J^P = \begin{bmatrix} J_v^P \\ J_\omega^P \end{bmatrix}$$

$$J_v^P = [R_0^0 k \times (p^0 - O_0^0), \; O_{3\times1}, \; O_{3\times1}]$$

$$J_\omega^P = [R_0^0 k, \; O_{3\times1}, \; O_{3\times1}]$$

② Computing Static Forces



Given $F_1, F_2, \ldots F_n$, compute the motor torque needed to keep the body from rotating.

Theory

Virtual work

$$\text{Work} = \sum F^T \cdot \delta r$$

← virtual displacement

1×2 2×1

1×1

$$\text{Work} = \tau^T \delta q$$

← virtual rotation

1×1 1×1

$$\tau^T \delta q = F^T \delta r$$

Divide by δq

$$\tau^T = F^T \underbrace{\left(\frac{\delta r}{\delta q}\right)}_{\simeq J_v}$$

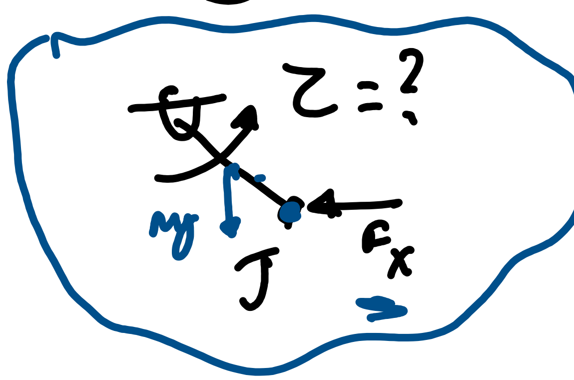
$$\tau^T = F^T J_v$$

Take transpose of both sides

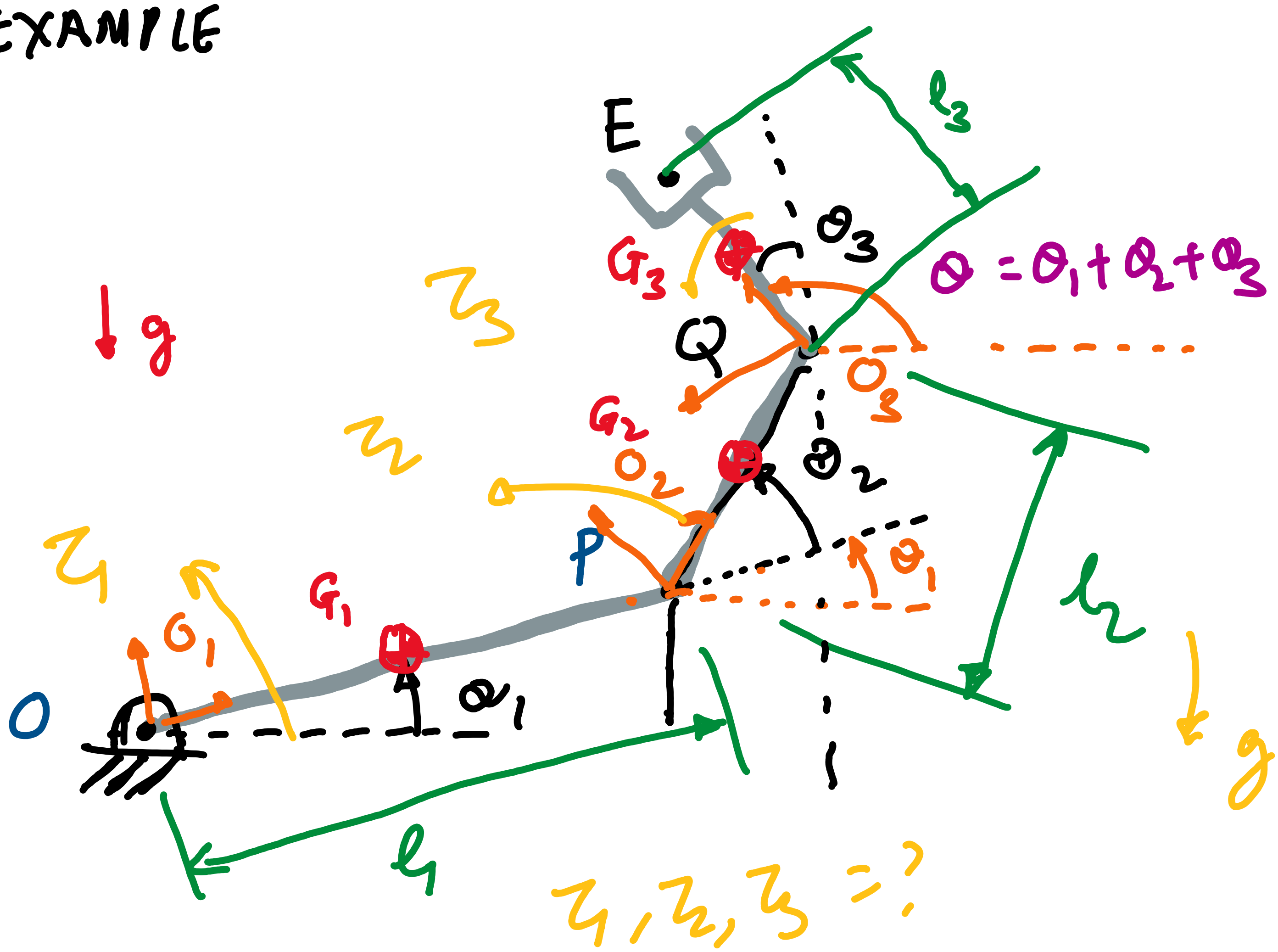
$$\tau = \left(F^T J_v\right)^T$$

$$= J_v^T F \qquad \left\{ (AB)^T = B^T A^T \right\}$$

$$\boxed{\tau = J_v^T F}$$



EXAMPLE



Compute the torques needed at each joint to hold the manipulator in equilibrium under gravity.

The masses are m_1 m_2 m_3 and the center of mass is located at midway of each link.

$$\tau = J_{v_{G_1}}^T F_{G_1} + J_{v_{G_2}}^T F_{G_2} + J_{v_{G_3}}^T F_{G_3}$$

$$\tau = \tau_1 + \tau_2 + \tau_3$$

$$F_{G_1} = \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} ; \quad F_{G_2} = \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix} ; \quad F_{G_3} = \begin{bmatrix} 0 \\ -m_3 g \\ 0 \end{bmatrix}$$

$$J_{v_{G_1}} = \begin{bmatrix} \underbrace{R_0^0 \hat{k} \times (g_1^0}_{\text{Revolute}} - o_1^0) & 0 & 0 \end{bmatrix}$$

$$J_{v_{G_2}} = \begin{bmatrix} R_0^0 \hat{k} \times (g_2^0 - o_1^0) & R_1^0 k \times (g_2^0 - o_2^0) & 0 \end{bmatrix}$$

$$J_{v_{G_3}} = \begin{bmatrix} R_0^0 k \times (g_3^0 - o_1^0) & R_1^0 k \times (g_3^0 - o_2^0) & R_2^0 k \times (g_3^0 - o_3^0) \end{bmatrix}$$

$$R_0^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2^0 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$O_1^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad O_2^0 = H_2^0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad O_3^0 = H_3^0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

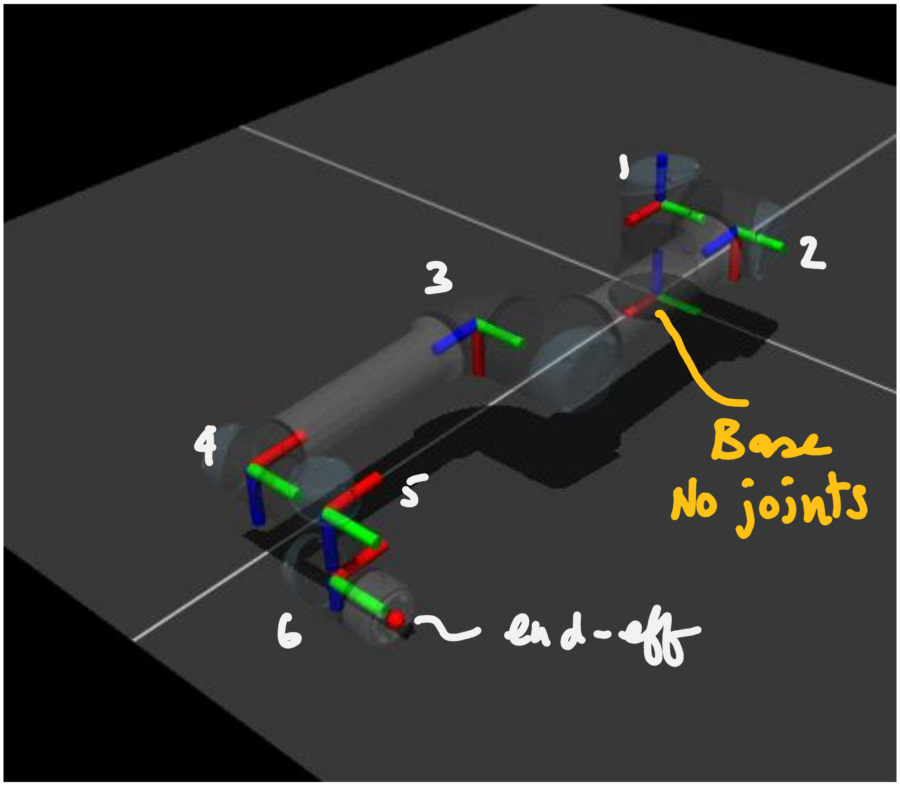
$$G_1^0 = H_1^0 \begin{bmatrix} l_1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad G_2^0 = H_2^0 \begin{bmatrix} l_2/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad G_3^0 = H_3^0 \begin{bmatrix} l_3/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tau_1 = \begin{bmatrix} -m_1 g\, l_1 c_1 /2 \\ 0 \\ 0 \end{bmatrix} \; ; \; \tau_2 = \begin{bmatrix} -m_2 g \left(l_1 c_1 + l_2 c_{12}/2 \right) \\ -m_2 g\, l_2\, c_{12}/2 \\ 0 \end{bmatrix}$$

$$\tau_3 = \begin{bmatrix} -m_3 g \left(l_1 c_1 + l_2 c_{12} + l_3 c_{123}/2 \right) \\ -m_3 g \left(l_2 c_{12}/2 + l_3 c_{123}/2 \right. \\ -m_3 g\, l_3\, c_{123}/2 \end{bmatrix}$$

and $\tau = \tau_1 + \tau_2 + \tau_3$

UR5 body & end-effector



① $J_{end\text{-}eff} = J_E$

② $J_{G_1}, J_{G_2}, J_{G_3}$
$J_{G_4}, J_{G_5}, J_{G_6}$