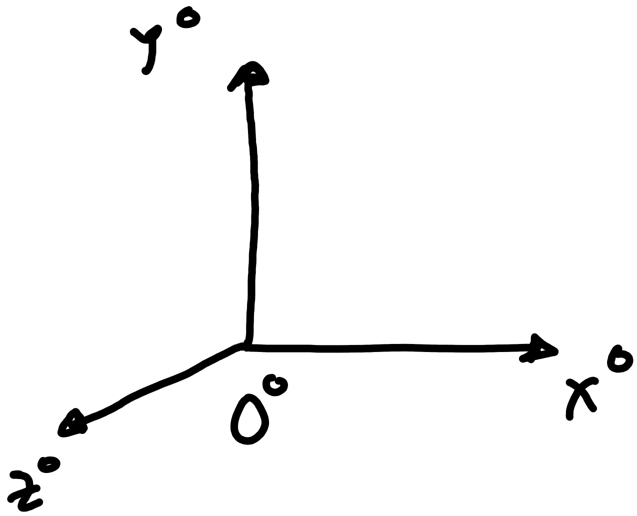
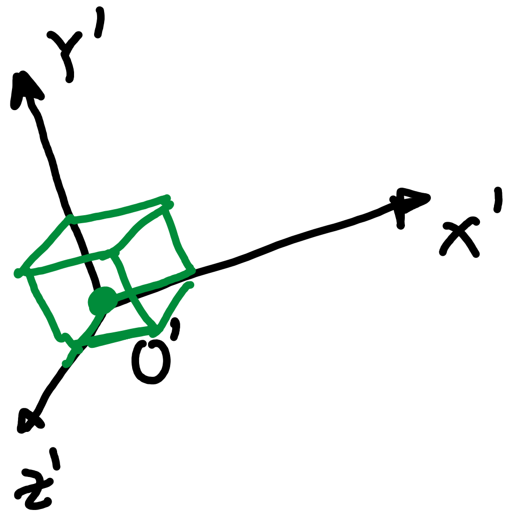
Coordinate Frames



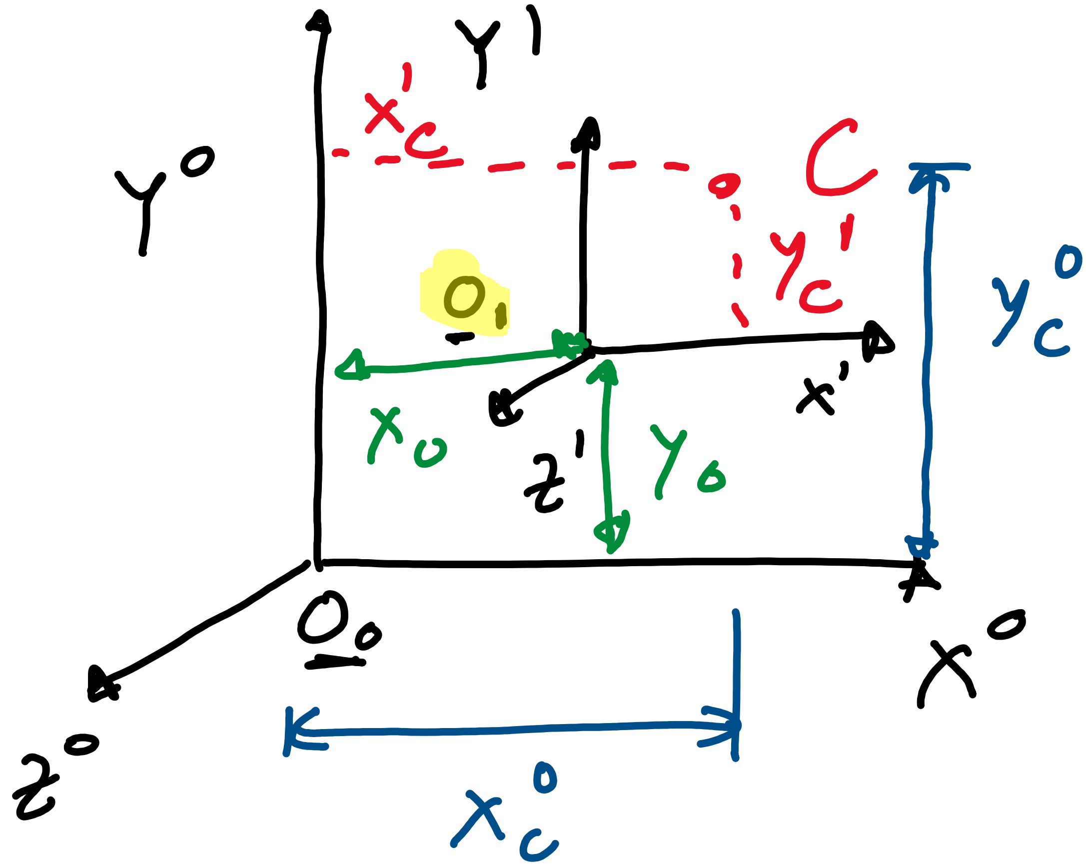


$i = 0$ World frame / Fixed Frame

$i = 1, 2, 3$ Body frame / Moving frame

attached to the body & moves with it.

Translation



$$C^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \\ z_c^0 \end{bmatrix} \qquad C^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \\ z_c^1 \end{bmatrix}$$

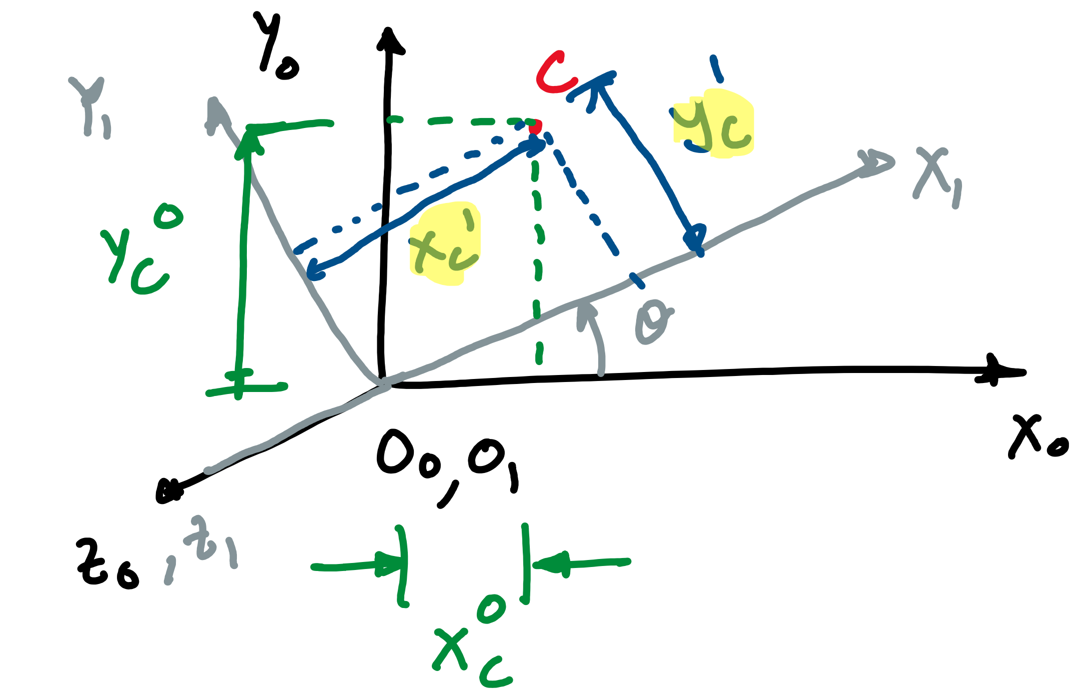
$$\underline{O}_1^0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}_{3\times 1} \qquad O_0^1 = \begin{bmatrix} -x_0 \\ -y_0 \\ -z_0 \end{bmatrix}$$

$\square \rightsquigarrow$ frame of reference

$- \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}_{3\times1} \quad \begin{bmatrix} -y_0 \\ -z_0 \end{bmatrix}$

$\rightsquigarrow$ frame of reference

Rotations



$$x_c^0 = \cos\theta \; x_c^1 - \sin\theta \; y_c^1$$

$$y_c^0 = \sin\theta \; x_c^1 + \cos\theta \; y_c^1$$

$$z_c^0 = z_c^1$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \\ z_c^0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c^1 \\ y_c^1 \\ z_c^1 \end{bmatrix}$$

$$C^0 = R_1^0 \; C^1$$

$$c^0 = R_1^0 \, c^1$$

$\downarrow$

Rotation of frame 1 with respect to frame 0

$$c^1 = (R_1^0)^{-1} \, c^0$$

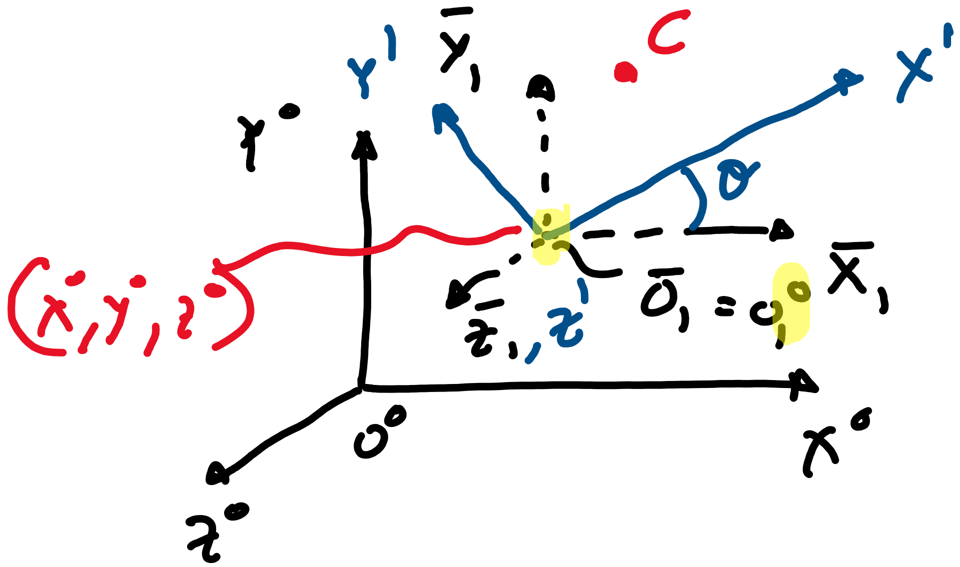
$$c^1 = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} c^0$$

$\rightarrow$ $$c^1 = (R_1^0)^T \, c^0$$

$\rightarrow$ $$\underline{c}^1 = R_0^1 \, \underline{c}^0$$

$$\boxed{(R_0^1) = (R_1^0)^T}$$

Combined Translation & Rotation



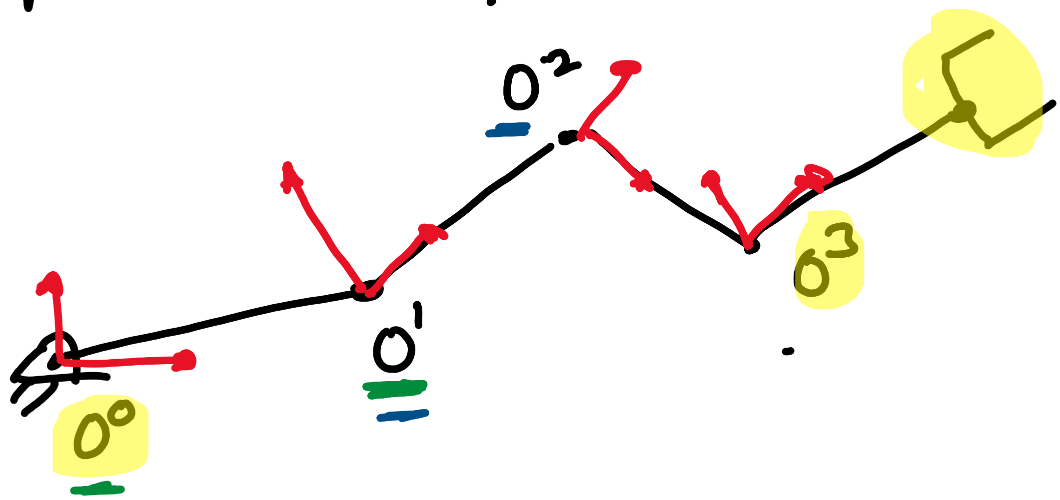
$$O^0 x^0 y^0 z^0 \longrightarrow \bar{O}^1 \bar{x}^1 \bar{y}^1 \bar{z}^1 \longrightarrow O^1 x^1 y^1 z^1$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \\ z_c^0 \end{bmatrix} = \begin{bmatrix} x^0 \\ y^0 \\ z^0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c^1 \\ y_c^1 \\ z_c^1 \end{bmatrix}$$

Translation Rotation

$$\boxed{C^0 = O_1^0 + R_1^0 \, C^1}$$

Multiple rotations / translation



$$O_0x_0y_0z_0 \longrightarrow O_1x_1y_1z_1 \longrightarrow O_2x_2y_2z_2 \cdots$$

$$\left.\begin{aligned} c^0 &= O_1^0 + R_1^0 c^1 \\ c^1 &= O_2^1 + R_2^1 c^2 \\ c^2 &= O_3^2 + R_3^2 c^3 \end{aligned}\right\}$$

$$c^0 = O_1^0 + R_1^0 \left(O_2^1 + R_2^1 c^2\right)$$

3x1

$$= O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 \left(O_3^2 + R_3^2 c^3\right)$$

$$\rightarrow\ c^0 = \underbrace{\underset{3\times1}{O_1^0} + \underset{3\times1}{R_1^0 O_2^1} + R_1^0 R_2^1 O_3^2}_{\text{Translation}} + \underbrace{R_1^0 R_2^1 R_3^2}_{\substack{3\times3 \\ \text{Rotation}}} \underset{3\times1}{c^3}$$

Homogenous Transformation

$$\underbrace{H_i^{i-1}}_{4\times 4} = \underbrace{\begin{bmatrix} \underbrace{R_i^{i-1}}_{3\times 3} & \underbrace{O_i^{i-1}}_{3\times 1} \\ \underbrace{[0\ 0\ 0]}_{1\times 3} & \underbrace{1}_{1\times 1} \end{bmatrix}}_{4\times 4}$$

$$\underline{c}^{i-1} = H_i^{i-1}\, \underline{c}^{i}$$

$$\underbrace{\begin{bmatrix} c^{i-1} \\ 1 \end{bmatrix}}_{4\times 1} = \underbrace{H_i^{i-1}}_{4\times 4} \underbrace{\begin{bmatrix} c^{i} \\ 1 \end{bmatrix}}_{4\times 1} \qquad c^i = \underbrace{\begin{bmatrix} x_c^i \\ y_c^i \\ z_c^i \end{bmatrix}}_{3\times 1}$$

e.g. $0 \longrightarrow 1$

$$O_0 x_0 y_0 z_0 \longrightarrow O_1 x_1 y_1 z_1$$

$$C^{i-1} = H_i^{i-1} C^i \qquad i = 1$$

$$C^0 = H_1^0 C^1$$

$$\begin{bmatrix} C^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C^1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \underset{3\times1}{C^0} \\ \underset{1\times1}{1} \end{bmatrix} = \begin{bmatrix} \underset{3\times1}{R_1^0 C^1 + O_1^0} \\ \underset{1\times1}{1} \end{bmatrix}$$

e.g. $0 \longrightarrow 1 \longrightarrow 2$

$$\left.\begin{aligned} C^0 &= H_1^0 \, C^1 \\ C^1 &= H_2^1 \, C^2 \end{aligned}\right\} \quad C^0 = H_1^0 \, H_2^1 \, C^2$$

$$\begin{bmatrix} C^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & O_2^1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C^2 \\ 1 \end{bmatrix}$$

$$= \underset{2\times 2}{\begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} R_2^1 C^2 + O_2^1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} C^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 C^2 + R_1^0 O_2^1 + O_1^0 \\ 1 \end{bmatrix}$$

$$C^0 = \underbrace{O_1^0 + R_1^0 O_2^1}_{\text{Translation}} + \underbrace{R_1^0 R_2^1}_{\text{Rotation}} C^2$$

$$H_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

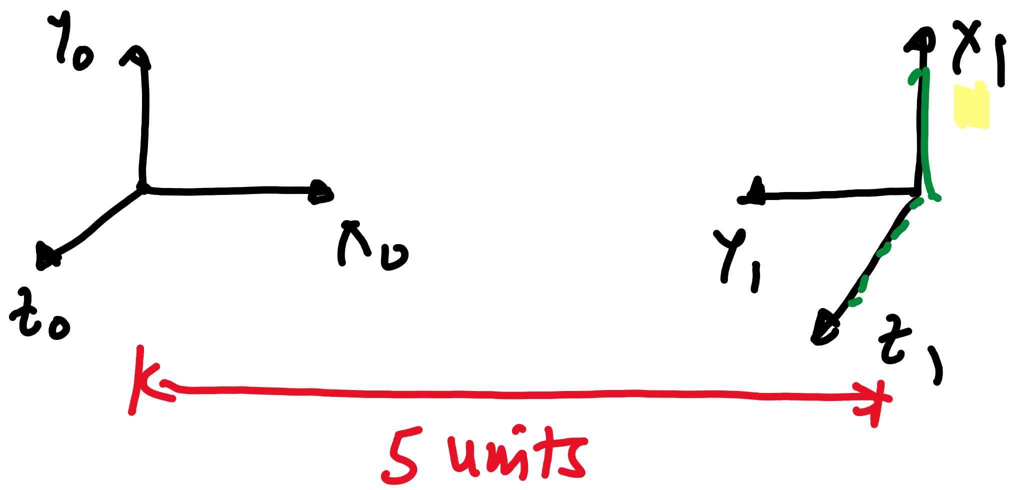
$$H_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_x(x) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_y(y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

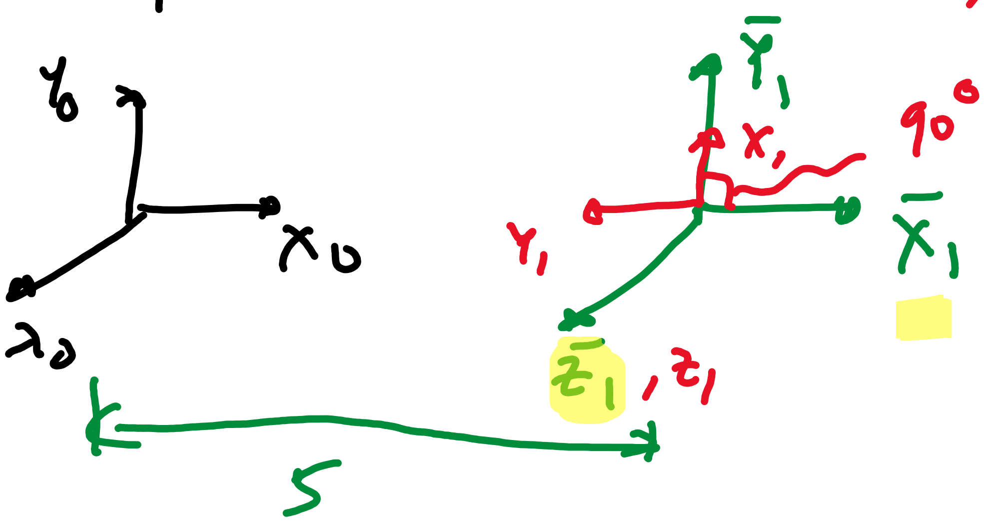
$$H_z(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE



Compute the Homogenous transformation that describes motion from frame 0 to frame 1

$$H_1^0 = H_x(5)\, H_z(\pi/2)$$



$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 & 0 & 0 \\ \sin \pi/2 & \cos \pi/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Answer)