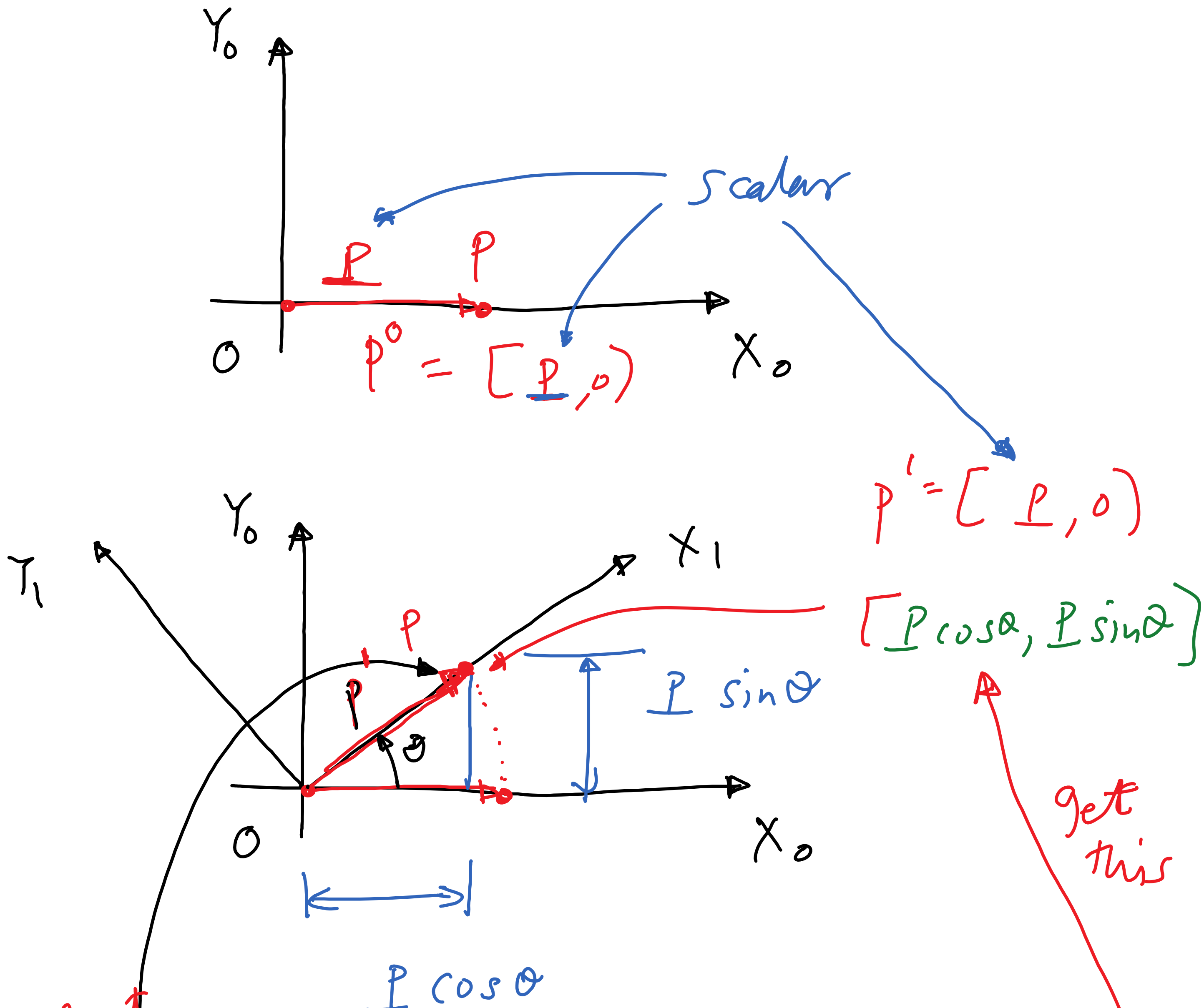
Rotation of a vector in coordinate frame



Important → $\boxed{p^0 = R_1^0 \, p^1}$ → I claim.

We will check this

$$p^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} p\cos\theta \\ p\sin\theta \end{bmatrix}$$

Multiple rotations

Consider rotation of a vector p from frame $o_0x_0y_0$ to $o_1x_1y_1$ to $o_2x_2y_2$

The vector has length p^0, p^1, p^2 in frames $o_0x_0y_0$, $o_1x_1y_1$, $o_2x_2y_2$ respectively

$$p^0 = R_1^0 p^1 \quad \text{— (1)}$$

$$p^1 = R_2^1 p^2 \quad \text{— (2)}$$

We want to find $\underline{\underline{R_2^0}}$

$$p^0 = R_2^0 p^2 \quad \text{— (3)}$$

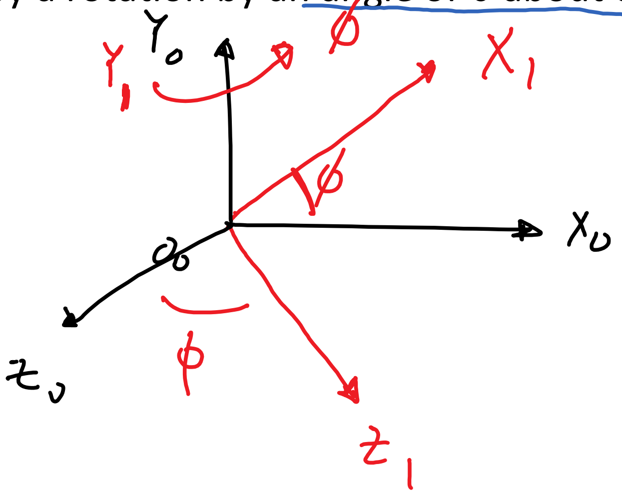
From (1) and (2)

$$p^0 = R_1^0 R_2^1 p^2 \quad \text{— (4)}$$

Compare (3) with (4)

$$\boxed{\underline{\underline{R_2^0}} = R_1^0 R_2^1}$$

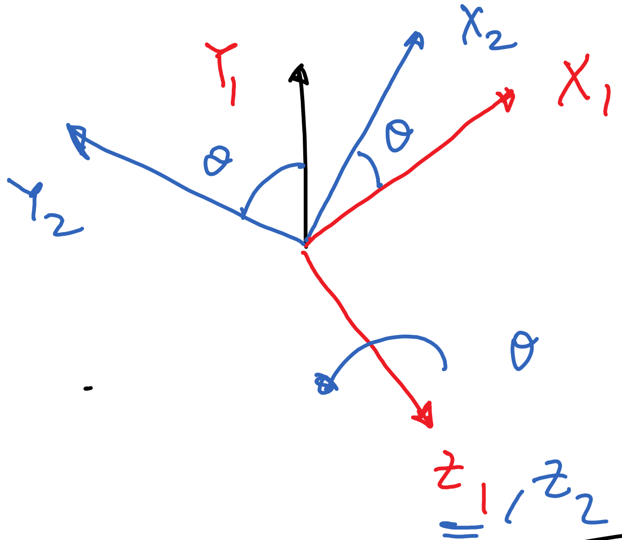
Example: If R represents rotation by an angle of Ø about current y-axis followed by a rotation by an angle of θ about current z-axis. Then find R



$$R_2^0 = R_1^0 \, R_2^1$$

$$\downarrow$$

$$R = R_{y,\phi} \, R_{z,\theta}$$



$c\phi = \cos\phi$

$s\theta = \sin\theta$

$$R = \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} c\phi \, c\theta & -c\phi \, s\theta & s\phi \\ s\theta & c\theta & 0 \\ -s\phi c\theta & s\phi \, s\theta & c\phi \end{bmatrix}$$

Example: If the above rotation is performed in reverse order find resulting matrix R'. That is, if R' represents rotation by an angle of Ø about current z-axis followed by a rotation by an angle of θ about current y-axis. Then find R

$$R' = \underline{R_{z,\phi} \; R_{y,\theta}} \qquad \{ R_2^0 = R_1^0 R_2^1 \}$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix}$$

$$R' = \begin{bmatrix} c\theta\, c\phi & -s\theta & c\theta\, s\phi \\ s\theta\, c\phi & c\theta & s\theta\, s\phi \\ -s\phi & 0 & c\phi \end{bmatrix}$$

$$R = \begin{bmatrix} c\phi\, c\theta & -c\phi\, s\theta & s\phi \\ s\theta & c\theta & 0 \\ -s\phi\, c\theta & s\phi\, s\theta & c\phi \end{bmatrix}$$ – From previous example

$R \neq R'$ Rotations do not commute

Example: We define R through the flowing step wise rotations. Find R

(1) α about current x-axis
(2) β about current z-axis
(3) γ about current y-axis
(4) δ about current x-axis

(1) $R = R_{x,\alpha}$

(2) $R = R_{x,\alpha} R_{z,\beta}$

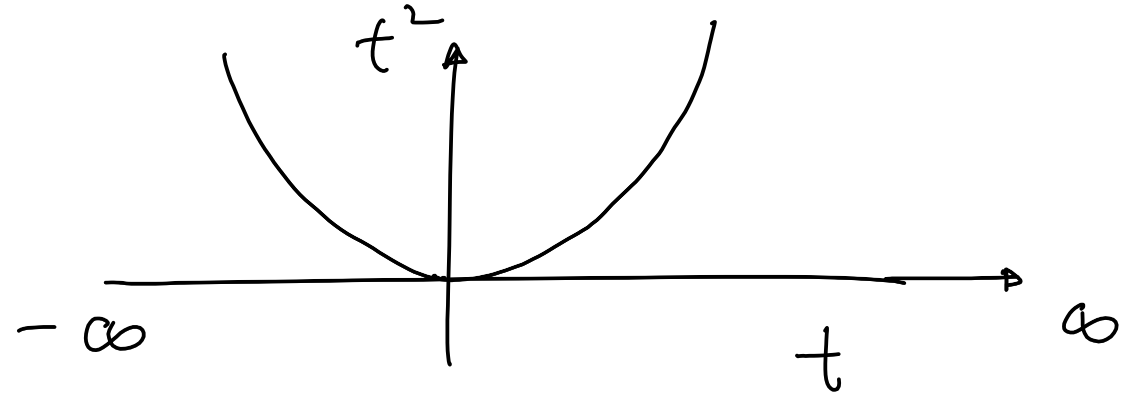
(3) $R = R_{x,\alpha} R_{z,\beta} R_{y,\gamma}$

(4) $\boxed{R = R_{x,\alpha} R_{z,\beta} R_{y,\gamma} R_{x,\delta}}$

Parameterization of rigid body rotation

Parameterization example $y = x^2$

Describe any point on the parabola as (t, t^2) (t is parameter)



Circle : $x^2 + y^2 = r^2$

Any point: $(r\cos\theta, r\sin\theta)$ $0 < \theta < 2\pi$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$ — 9 numbers

Does one need 9 numbers to describe rotations

Property $RR^T = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (constrains the 9 numbers)

these constrains are

$$\left.\begin{aligned}
r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\
r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\
r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1 \\
r_{11} r_{12} + r_{21} r_{22} + r_{31} r_{32} &= 0 \\
r_{12} r_{13} + r_{22} r_{23} + r_{32} r_{33} &= 0 \\
r_{13} r_{11} + r_{23} r_{21} + r_{33} r_{31} &= 0
\end{aligned}\right\} \text{6 conditions}$$

$$9 - 6 = 3$$

unique parameters needed to describe rotation.

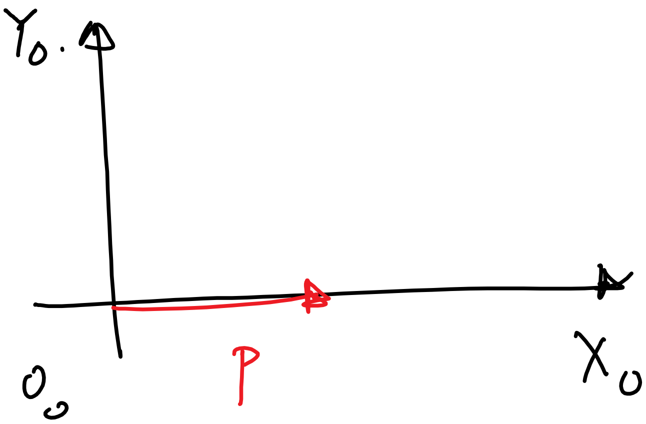
We will use 3 rotations to paramlterize rotations

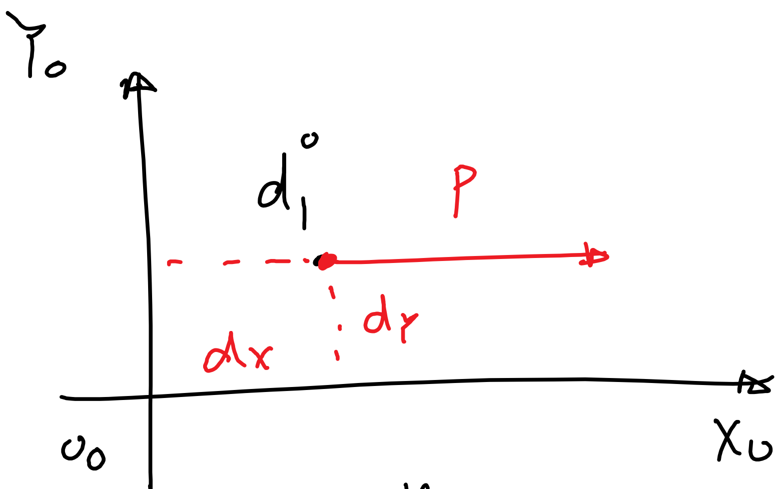
12 representations

XYZ	YZX	ZXY
XZY	YXZ	ZYX
XYX	YXY	ZYZ
XZX	YZY	ZXZ

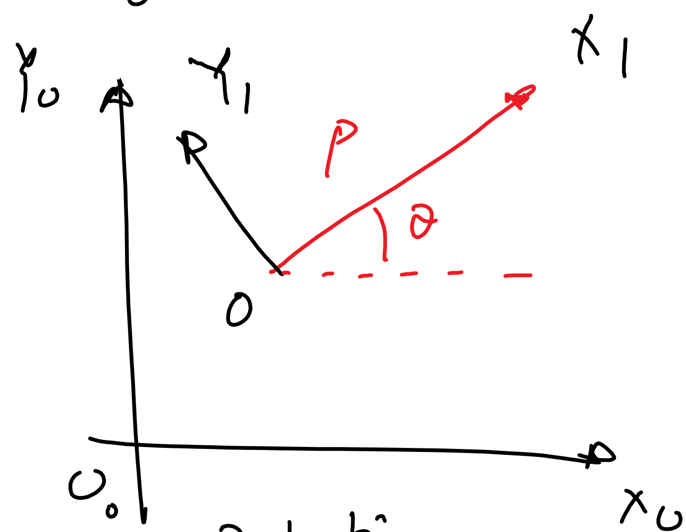
Euler angles

Rigid motions





Translation



Rotation

① $-\ \underline{p}^0 = d_1^0 + R_1^0\, p^1 \quad (O_0 X_0 Y_0 \to O_1 X_1 Y_1)$

② $-\ \underline{p}^1 = d_2^1 + R_2^1\, \underline{p}^2 \quad (O_1 X_1 Y_1 \to O_2 X_2 Y_2)$

Put ② in ①

$$p^0 = d_1^0 + R_1^0 (\underbrace{d_2^1 + R_2^1 p^2}_{p^1})$$

$$\Rightarrow \quad p^0 = d_1^0 + R_1^0 d_2^1 + R_1^0 R_2^1 p^2$$

n translation + rotation.

$$\Rightarrow \quad p^0 = \underline{d_1^0 + R_1^0 d_2^1 + R_1^0 R_2^1 d_3^2 + \ldots R_1^0 R_2^1 \ldots R_n^{n-1} p^n}$$

very messy if there are multiple movements.

A compact way of representing rotations and translation is through a <u>homogenous transformation</u>

Homogenous transformation (H)

$$H = \begin{bmatrix} R_{3\times3} & d_{3\times1} \\ 0_{1\times3} & 1_{1\times1} \end{bmatrix}_{4\times4}$$

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

$$p^0 = H_1^0 p^1$$

$$p^1 = H_2^1 p^2$$

$$p^0 = H_1^0 H_2^1 p^2 \quad \text{(subs)}$$

$$p^0 = H_1^0 \; H_2^1 \; p^2$$

$$= \underset{4\times4}{\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}} \underset{4\times4}{\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix}} p^2$$

$$\underset{4\times1}{\begin{bmatrix} p^0 \\ 1 \end{bmatrix}} = \underset{4\times4}{\begin{bmatrix} R_1^0 R_1^2 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} p^2 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_1^2 p^2 + R_1^0 d_2^1 + d_1^0 \\ 0 \qquad\qquad 1 \end{bmatrix}$$

$$p^0 = R_1^0 R_1^2 p^2 + R_1^0 d_2^1 + d_1^0$$

$$\boxed{p^0 = d_1^0 + R_1^0 d_2^1 + R_1^0 R_2^1 p^2 \quad \text{— from previous pages}}$$

$$\boxed{p^0 = H_1^0 \; H_2^1 \; H_3^2 \; \cdots \; H_n^{n-1} \; p^n}$$

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c\beta & 0 & s\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s\beta & 0 & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c\gamma & -s\gamma & 0 & 0 \\ s\gamma & c\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Find the homogenous transformation H that results from the following step wise transformation

(1) rotation by an angle α about current x-axis

(2) translation of b units along current x-axis

(3) translation of d units along current z-axis

(4) rotation by an angle of θ about current z-axis

(1) $H = Rot_{x,\alpha}$

(2) $H = Rot_{x,\alpha}\, Trans_{x,b}$

(3) $H = Rot_{x,\alpha}\, Trans_{x,b}\, Trans_{z,d}$

(4) $\boxed{\underset{4\times4}{H} = Rot_{x,\alpha}\, Trans_{x,b}\, Trans_{z,d}\, Rot_{z,\theta}}$