Example 2: Find a time based parameterization for a revolute joint of a manipulator. The joint should move from 0 to 0.5 rad from time t= 0 to t=1 sec followed by movement from 0.5 rad to 1 rad ~~is~~ from t=1 to t=3 secs. Also, the velocity of the joint at the start of motion (t=0) and end of motion (t=3) should be 0 and the acceleration of the joint at the intermediate point (t=1) should be continuous. Assume two minimal order polynomials of time, one for each movement.

$\rightarrow q_1(0) = 0 \; ; \; q_1(1) = 0.5 \; ; \; q_2(1) = 0.5 \; ; \; q_2(3) = 1$

$\rightarrow \dot{q}_1(0) = 0 \; ; \; \dot{q}_2(3) = 0 \; ; \; \ddot{q}_1(1) = \ddot{q}_2(1)$

$\rightarrow \dot{q}_1(1) = \dot{q}_2(1)$

8 conditions

$$q_1(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$q_2(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$$\dot{q}_1(t) = a_{11} + 2 a_{12} t + 3 a_{13} t^2$$

$$\dot{q}_2(t) = a_{21} + 2 a_{22} t + 3 a_{23} t^2$$

$$\ddot{q}_1(t) = 2 a_{12} + 6 a_{13} t$$

$$\ddot{q}_2(t) = 2 a_{22} t + 6 a_{23} t$$

$\rightarrow$ subs 8 conditions on q_1 and q_2 to get 8 equations.

$$\underline{q_1(0) = 0} = a_{10} \quad \text{— I}$$

$$q_1(1) = 0.5 = a_{10} + a_{11} + a_{12} + a_{13} \quad \text{— II}$$

$$q_2(1) = 0.5 = a_{20} + a_{21} + a_{22} + a_{23} \quad \text{— III}$$

$$q_2(3) = 1 = a_{20} + 3a_{21} + 9a_{22} + 27a_{23} \quad \text{— IV}$$

$$\dot{q}_1(0) = 0 = a_{11} \quad \text{— V}$$

$$\dot{q}_2(3) = 0 = a_{21} + 6a_{22} + 27a_{23} \quad \text{— VI}$$

$$\ddot{q}_1(1) - \ddot{q}_2(1) = a_{12} + 3a_{13} - a_{22} - 3a_{23} = 0 \quad \text{— VII}$$

$$\dot{q}(1) - \dot{q}_2(1) = a_{11} + 2a_{12} + 3a_{13} - a_{21} - 2a_{22} - 3a_{23} = 0 \quad \text{— VIII}$$

Put I — VIII in matrix

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\ 0 & 0 & 1 & 3 & 0 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 & 0 & -1 & -2 & -3 \end{bmatrix}}_{A_{8\times 8}} \underbrace{\begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}}_{X_{8\times 1}} = \underbrace{\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_{8\times 1}}$$

$$X = \text{inv}(A)\, b = A \backslash b$$

$\text{inv}(A)$

$a_{10} = 0$; $a_{11} = 0$; $a_{12} = 0.875$ $a_{13} = -0.375$

$a_{20} = -0.4063$, $a_{21} = 1.2188$; $a_{22} = -0.3438$

$a_{23} = 0.0312$

For plots q vs. t, $\dot{q}$ vs. t, $\ddot{q}$ vs. t — see MATLAB

These ideas can be extended to end-effector trajectory generation.

1. Define conditions of $x(t), y(t) \ldots$

   e.g. $x(0) = 0$, $x(1) = 0.5$

   $\dot{x}(0) = \dot{x}(1) = 0$ $\quad \dot{y}(0) = \dot{y}(1) = 0$

2. Find $x(t)$, $y(t)$

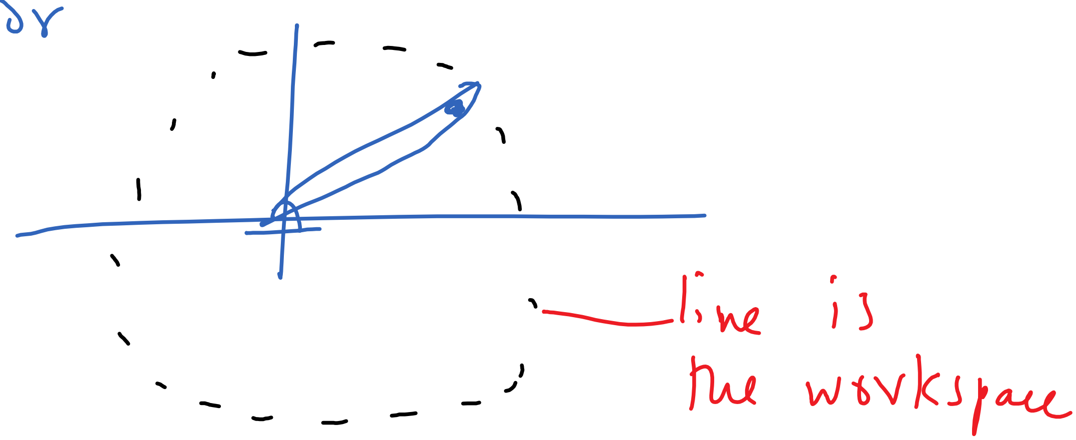
3. Use inverse kinematics to solve for $\theta(t)$ or $d(t)$

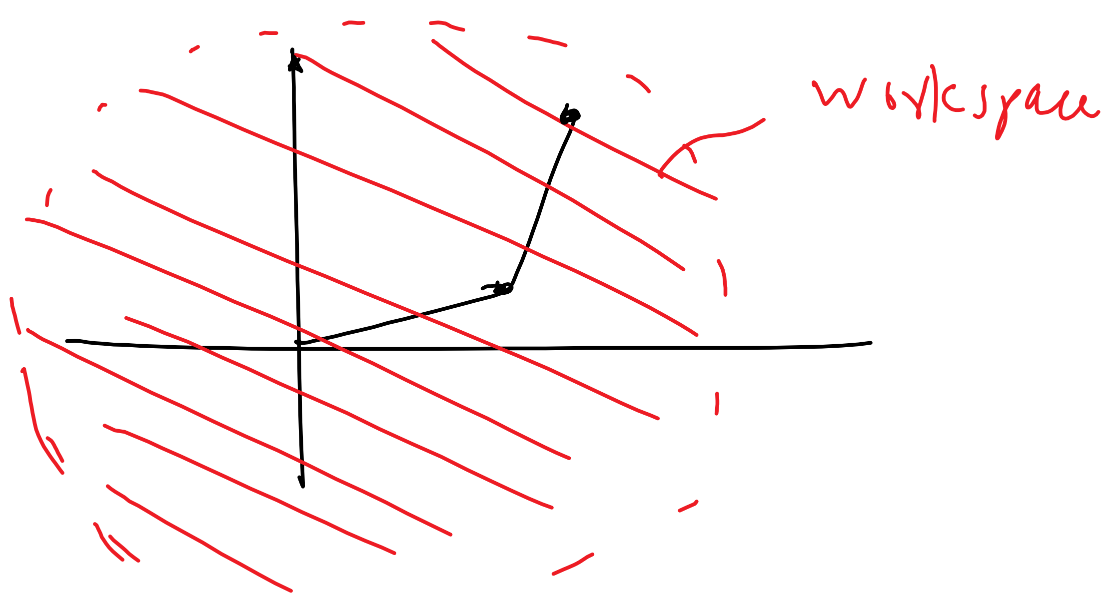
Comments about inverse kinematics

→ given x, y solve for α_1, α_2.

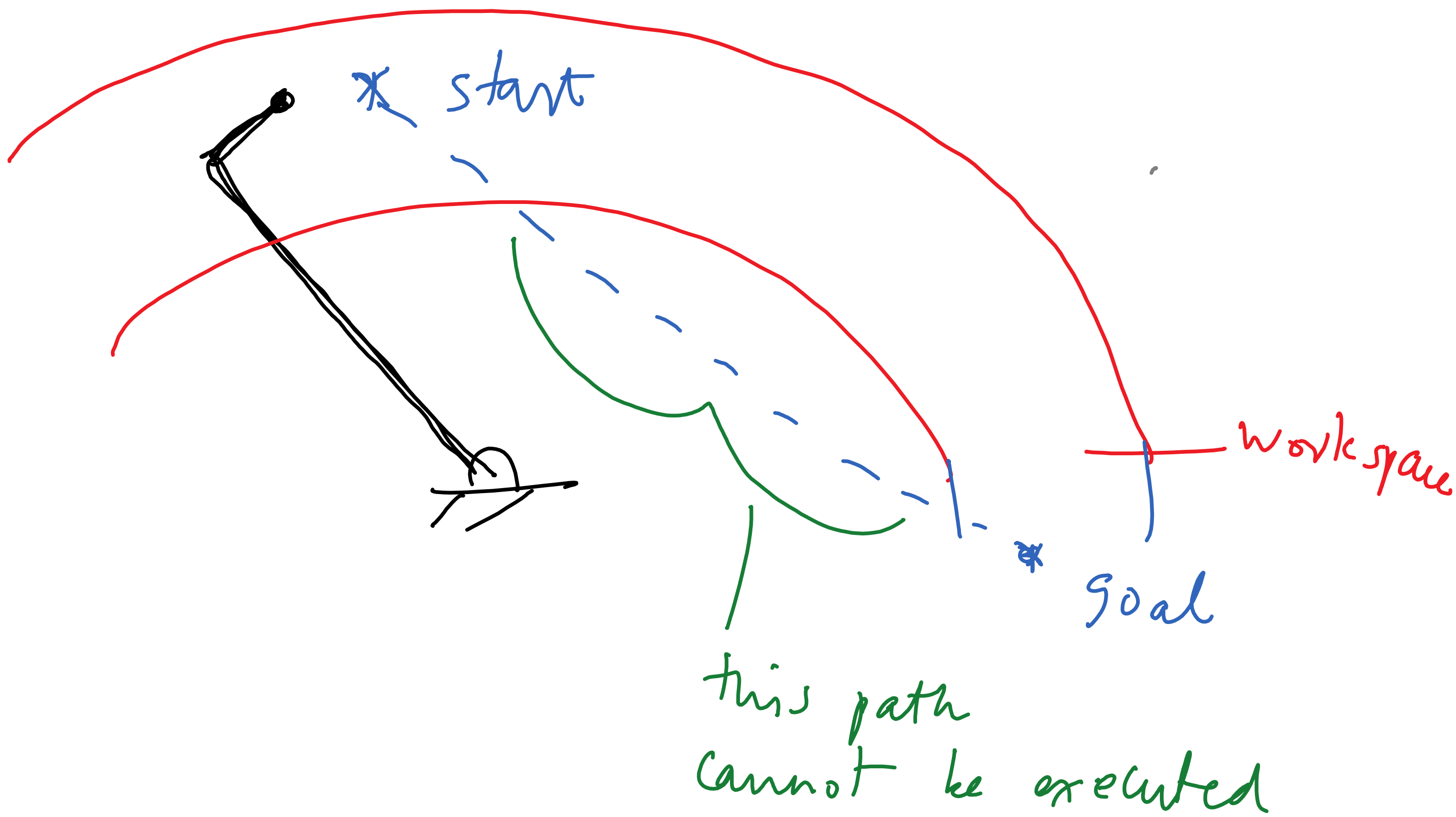
① Ensure that the positions x, y is reachable by the manipulator

Find the reachable space of the manipulator

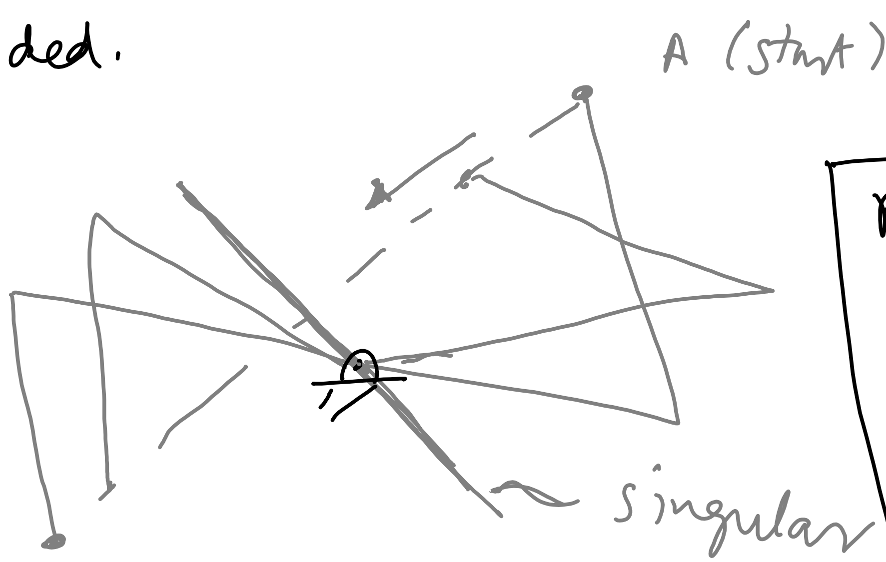




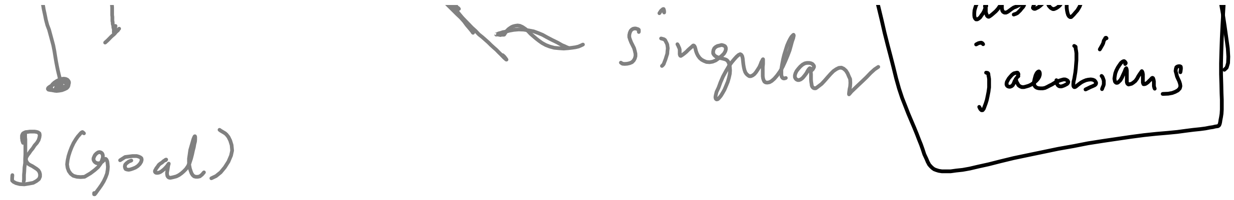
② End points are reachable but intermediate points are not reachable



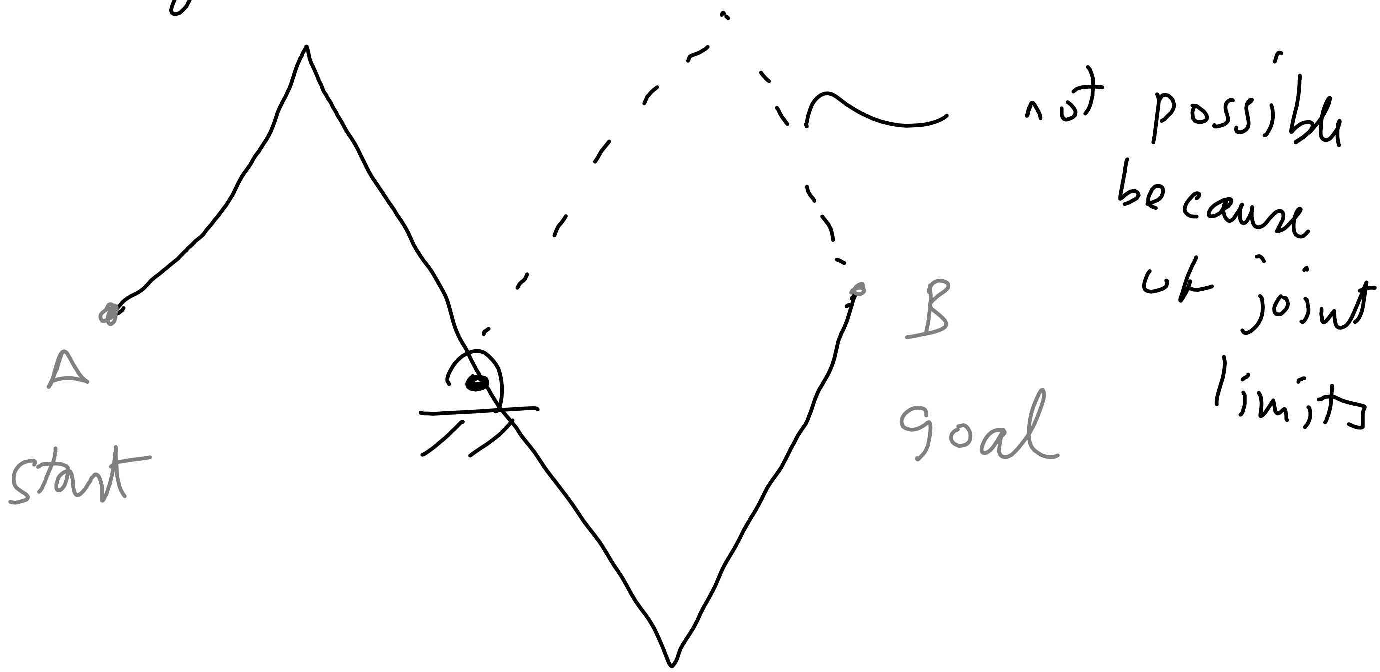
③ Singular configurations should be avoided.



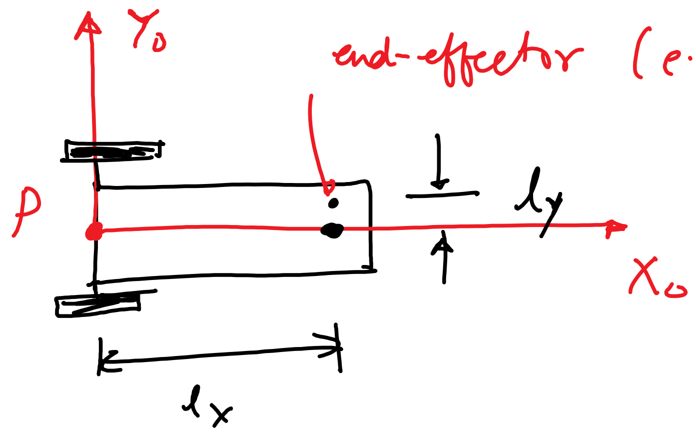
Revisit when we talk about jacobians


B (goal)
singular
jacobians

(3) start and end points are reached by different solutions



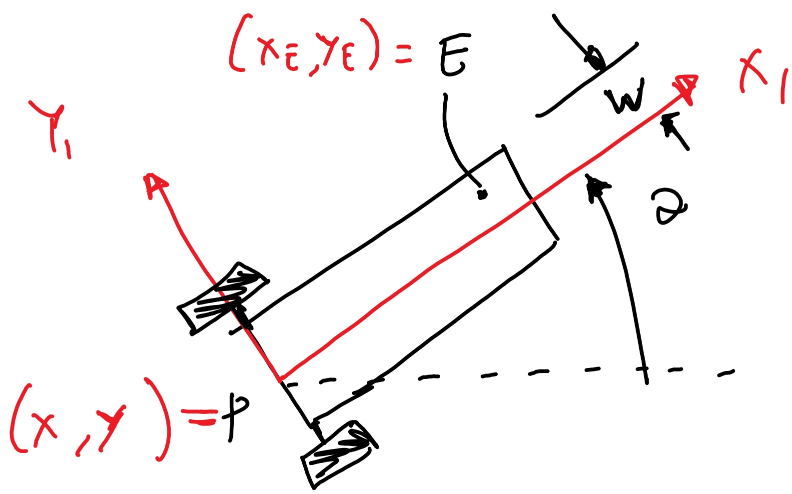
Inverse kinematics of a differential drive car



$X_0 Y_0$ - global frame

$X_1 Y_1$ - local frame attached to the car and moves with the car.

At a new time t



$$\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} l_x \\ l_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \quad - \text{I}$$

In a previous class we wrote equations for a diff-drive car

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2}(\dot{\phi}_r + \dot{\phi}_l)\cos\theta \\ \frac{r}{2}(\dot{\phi}_r + \dot{\phi}_l)\sin\theta \\ \frac{r}{2w}(\dot{\phi}_r - \dot{\phi}_l) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \underline{\dot{\theta}} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \underline{\omega} \end{bmatrix} \quad \textcircled{\text{II}}$$

where

$$v = \frac{r}{2}(\dot{\phi}_r + \dot{\phi}_l)$$

$$\omega = \frac{r}{2w}(\dot{\phi}_r - \dot{\phi}_l)$$

Diff. (I) with respect to time

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \end{bmatrix} = \begin{bmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\underbrace{\dot{\theta}}_{\omega} & -\sin\theta\,\dot{\theta} \end{bmatrix} \begin{bmatrix} l_x \\ l_y \end{bmatrix} + \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\overbrace{}^{\text{from (II)}}$$

<u>Re-arrange</u>: put $\dot{\theta} = \omega$, $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \end{bmatrix} = \begin{bmatrix} (-l_x \sin\theta - l_y \cos\theta)\,\omega \\ (l_x \cos\theta - l_y \sin\theta)\,\omega \end{bmatrix} + \begin{bmatrix} V\cos\theta \\ V\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \end{bmatrix} = \begin{bmatrix} \cos\theta & (-l_x \sin\theta - l_y \cos\theta) \\ \sin\theta & l_x \cos\theta - l_y \sin\theta \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix} \quad \text{— III}$$

Assume a proportional controller

$$\Rightarrow \dot{x}_E = k_p (x_{ref} - x_c) \qquad \dot{y}_E = k_p (y_{ref} - y_c) \quad \text{— IV}$$

k_p – use defined constant

x_{ref}, y_{ref} – target motion

x_c, y_c – current position of the point E

Put IV in III & solve for $\begin{pmatrix} v \\ \omega \end{pmatrix}$

$$\boxed{\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta - \left(\frac{l_y}{l_x}\right)\sin\theta & \sin\theta + \left(\frac{l_y}{l_x}\right)\cos\theta \\ -\left(\frac{1}{l_x}\right)\sin\theta & \left(\frac{1}{l_x}\right)\cos\theta \end{bmatrix} \begin{bmatrix} k_p(x_{ref} - x_c) \\ k_p(y_{ref} - y_c) \end{bmatrix}}$$

solve for v, ω at every time step

$$\dot{\phi}_r = \frac{v + \omega w}{r} \qquad \dot{\phi}_l = \frac{v - \omega w}{r}$$

solve for $\dot{\phi}_r$, $\dot{\phi}_l$ given v, ω at every step.