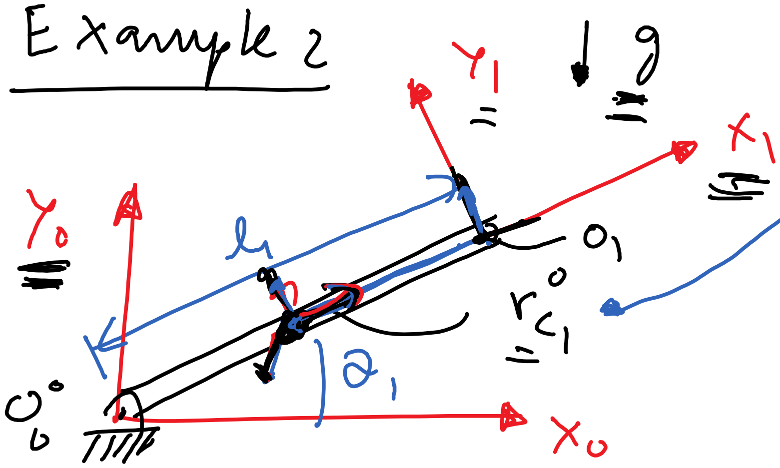
Example 2



$$r_{c_1} = A_1^0 \begin{pmatrix} -l_1/2 \\ 0 \\ 0 \end{pmatrix}$$

Link i	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1

Question:	For the one-link manipulator with center of mass midway as shown, find the equation of motion. Assume that the mass of the link is m1, inertia about the principle axis is I1

④ $L = K - P$

$$K = \frac{1}{2}\,\dot{q}^T \left\{ m_1 J_{v_{c_1}}^T J_{v_{c_1}} + J_{\omega_{c_1}}^T R_{b_1} I_1 R_{b_1}^T J_{\omega_{c_1}} \right\} \dot{q}$$

$$P = \underset{\uparrow}{-}\, m_1 g^T r_{c_1}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

because base frame is same as 1

$$\dot{q} = \dot{\theta}_1$$

③ Remind: Revolute:

$$J_{v_{c_i}} = R_{i-1}^0 k \times (O_{c_i}^0 - O_0^0)$$

$$J_{\omega_{c_i}} = R_{i-1}^0 k$$

$$J_{v_{c_1}} = R_b^0 k \times (r_{c_1}^0 - \underbrace{O_b^0}_{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}})$$

$$J_{\omega_{c_1}} = R_b^0 k$$

$$A_b^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_b^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\checkmark\ \underline{\underline{R_0^0 k}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_0^0} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\hat{k}} = \begin{bmatrix} 0 \\ 0 \\ \underline{1} \end{bmatrix}$$

$$r_{c_1}^0 = A_1^0 \begin{pmatrix} -l_1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -l_1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1 = \cos\theta_1 \qquad s_1 = \sin\theta_1$$

$$\underline{\underline{r_{c_1}^0}} = \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{v_{c_1}} = R_0^0 k \times \left(r_{c_1}^0 - o_b^0 \right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{pmatrix}$$

$$J_{v_{c_1}} = \begin{bmatrix} 0 & \overset{-q_z}{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1/2 \\ l_1 c_1/2 \\ 0 \end{bmatrix}$$

cross((0 0 1), l_1 . . .)

$$K = \frac{1}{2}\dot{\theta}\left\{ \overbrace{m_1 \begin{bmatrix} -\frac{l_1 c_1}{2} & \frac{l_1 s_1}{2} & 0 \end{bmatrix} \begin{bmatrix} -l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{bmatrix}}^{m\, J_v^T J_v} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} I_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{J_\omega^T I_1 J_\omega} \right\} \dot{\theta}$$

$$I_1 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$= \frac{1}{2}\dot{\theta}\left\{ \frac{m_1 l_1^2}{4} + I_z \right\}\dot{\theta}$$

$$K = \frac{1}{2}\left(\frac{m_1 l_1^2}{4} + I_z \right)\dot{\theta}^2$$

$$P = -m_1 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{bmatrix} = m_1 g \frac{l_1 s_1}{2}$$

$$\mathcal{L} = K - P$$

$$\mathcal{L} = \frac{1}{2}\left(\frac{m_1 l_1^2}{4} + I_z\right)\dot{\theta}_1^2 - m_1 g l_1 \frac{\sin\theta_1}{2}$$

(5) Equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k$$

$$q_k = \theta_1$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = \tau_1 = T_1$$

$$\Rightarrow \frac{d}{dt}\left(\left(\frac{m_1 l_1^2}{4} + I_z\right)\dot{\theta}_1\right) - \frac{m_1 g l_1 \cos\theta_1}{2} = T_1$$

$$\Rightarrow \boxed{\left(\frac{m_1 l_1^2}{4} + I_z\right)\ddot{\theta}_1 - m_1 g \frac{l_1 \cos\theta_1}{2} = T_1}$$

Solving for $$\boxed{\ddot{\theta}_1 = \frac{1}{\left(I_z + \frac{m l_1^2}{4}\right)} \left\{ T - \frac{m_1 g l_1 \cos\theta_1}{2} \right\}}$$ I

We will simulate the system in MATLAB

1) Initial state

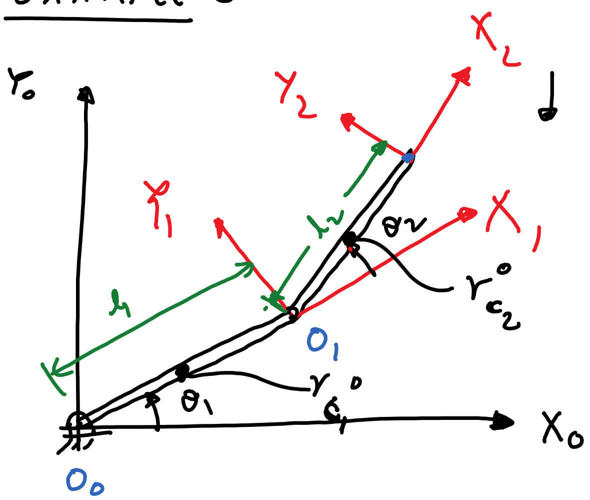
$\theta_1(t=0) = ?$

$\dot{\theta}_1(t=0) = ?$

2) Use ode 4 or ode 45 to integrate I

See MATLAB code.

EXAMPLE 3



D-H Table

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2

Question: For the two-link manipulator shown above find the equations of motion. Assume that the center of mass for each link is midway of the link, the masses are m1 and m2, and inertia about the principle axis is I1 and I2.

$$K = \frac{1}{2}\dot{q}^T \Big\{ m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} \dots$$

$$+ J_{\omega_{c_1}}^T R_b^1 I_1 (R_b^1)^T J_{\omega_{c_1}} + \dots$$

$$+ J_{\omega_{c_2}} R_b^2 I_2 (R_b^2)^T J_{\omega_{c_2}} \Big\} \dot{q}$$

$$P = -m_1 g^T r_{c_1}^0 - m_1 g^T r_{c_2}^0$$

$$\dot{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\boxed{J_{v_{c_1}}, J_{v_{c_2}}, J_{\omega_{c_1}}, J_{\omega_{c_2}}, r_{c_1}^0, r_{c_2}^0 = ?}$$

$$R_b^1 = R_b^2 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Update later

$\mathcal{L} = K - P$ is too complex for hand calculation. We will use MATLAB.

Simple example on using MATLAB to derive the equations of motion

$$K = \frac{1}{2} m \dot{y}^2$$

$$P = -m(-g)\,y = mgy$$

$$\mathcal{L} = \frac{1}{2} m \underline{\underline{\dot{y}}}^2 - mgy$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = f$$

$$\frac{d}{dt}\left(m\dot{y}\right) - \left(-mg\right) = f$$

$$\boxed{m\ddot{y} + mg = f}$$

Using symbolic tool box

syms m g f y ydot y2dot real

K = 0.5 * m * ydot * ydot;

P = m * g * y;

* L = K - P;

dLdydot = diff(L, ydot); $\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = f \right]$

= $m\dot{y}$

dLdy = diff(L, y);

ddt_dldydot = diff(~~dLdydot~~, t) will not work because t is not defined

??

I

explained next

$\boxed{\text{ddt\_dLdydot = diff(dLdydot, y) ydot + ... diff(dLdydot, ydot) y2dot;}}$

EOM = ddt - dLdydot - dLdy - f

run the code :

EOM = mg + my2dot - f = 0

Explanation for II

$\Rightarrow \frac{d}{dt} \cos(t) \Rightarrow$ diff(cos(t), t) = $-\sin t$

Explicit differentiation

$\Rightarrow F = \cos(q(t))$; Find $\frac{d}{dt}(\cos(q(t)))$

$F = \cos(q) \longrightarrow q \longrightarrow t$

$$\frac{dF}{dt} = \frac{dF}{dq}\frac{dq}{dt} = \frac{d}{dq}(\cos(q(t)))\frac{dq}{dt}$$

$$= (-\sin q)\,\dot{q}$$

Implicit differentiation

Back to diff. of lagrangian

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)} = \frac{d}{dq_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial q_k}{dt} +$$

$$\frac{d}{d\dot{q}_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\frac{\partial \dot{q}_k}{dt}$$

$$= \mathrm{diff}\left(\frac{\partial L}{\partial \dot{q}_k}, q_k\right)\dot{q}_k +$$

$$\mathrm{diff}\left(\frac{\partial L}{\partial \dot{q}_k}, \dot{q}_k\right)\ddot{q}_k$$