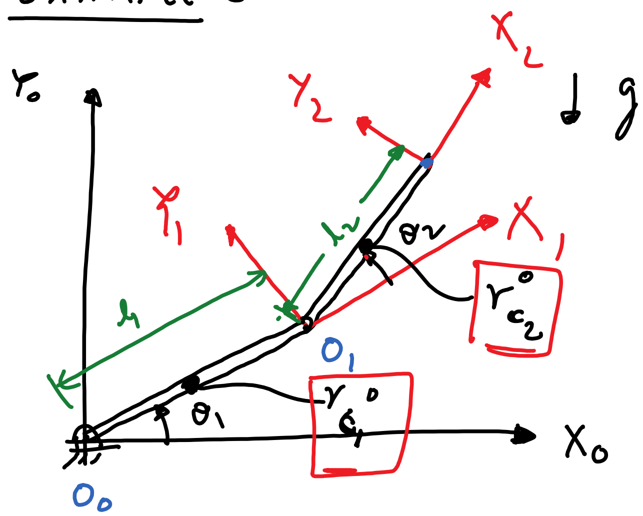
EXAMPLE 3



D-H Table

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2

Question: For the two-link manipulator shown above find the equations of motion. Assume that the center of mass for each link is midway of the link, the masses are m1 and m2, and inertia about the principle axis is I1 and I2.

$$\Rightarrow K = \frac{1}{2}\dot{q}^T \Big\{ m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} \ldots$$

$$+ J_{\omega_{c_1}}^T R_b^1 I_1 (R_b^1)^T J_{\omega_{c_1}} + \ldots$$

$$+ J_{\omega_{c_2}} R_b^2 I_2 (R_b^2)^T J_{\omega_{c_2}} \Big\} \dot{q}$$

Scalar

$\dot{q}^T = [\dot{\theta}_1 \;\; \dot{\theta}_2]$ 1×2

$\dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$ 2×1

$$P = -m_1 g^T r^0_{c_1} - m_1 g^T r^0_{c_2}$$

Scalar

$$\dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\boxed{J_{v_{c_1}}, J_{v_{c_2}}, J_{\omega_{c_1}}, J_{\omega_{c_2}}, r^0_{c_1}, r^0_{c_2} = ?}$$

Update later

$$R_b^1 = R_b^2 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{L} = K - P$ is too complex for hand calculation. We will use MATLAB.

$$A_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^0 = A_1^0 A_2^1 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c_1 = \cos\theta_1$; $s_1 = \sin\theta_1$; $c_2 = \cos\theta_2$; $s_2 = \sin\theta_2$

$c_{12} = \cos(\theta_1 + \theta_2)$; $s_{12} = \sin(\theta_1 + \theta_2)$

Revolute joint: $J_{v_i} = R_{i-1}^0 \, k \times (O_n^0 - O_{i-1}^0)$

$J_{\omega_i} = R_{i-1}^0 \, k$

$k = [0 \;\; 0 \;\; 1]^T$

$$r_{c_1} = A_1^0 \begin{bmatrix} -l_1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$r_{c_2} = A_2^0 \begin{bmatrix} -l_2/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 c_1 + l_2 c_{12}/2 \\ l_1 s_1 + l_2 s_{12}/2 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{v_{c_1}} = \left[\underline{R_0^0 k \times (r_{c_1}^0 - o_0^0)}_{3\times1} \quad \underline{0}_{3\times1} \right]$$

Revolute joint: $J_{v_i} = R_{i-1}^0 k \times (o_n^0 - o_{i-1}^0)$ ← point of interest

$J_{\omega_i} = R_{i-1}^0 k$

$k = [0 \ 0 \ 1]^T$

$$R_0^0 k \times (r_{c_1}^0 - o_0^0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{bmatrix}$$

$$A_0^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

use cross $= \begin{bmatrix} -l_1 s_1/2 \\ l_1 c_1/2 \\ 0 \end{bmatrix}$

$$J_{v_{c_1}} = \begin{bmatrix} -l_1 s_1/2 & 0 \\ l_1 c_1/2 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}$$

$$J_{v_{c_2}} = \left[\underbrace{R_0^0 k \times (r_{c_2}^0 - o_0^0)}_{3\times1} \quad \underbrace{R_1^0 k \times (r_{c_2}^0 - o_1^0)}_{3\times1} \right]$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_{12}/2 & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12}/2 & l_2 c_{12} \\ 0 & 0 \end{bmatrix}_{3\times2}$$

$$J_{\omega_{c_1}} = \begin{bmatrix} R_0^0 k_{3\times 1} & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$R_0^0 k \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J_{\omega_{c_2}} = \begin{bmatrix} R_0^0 k & R_1^0 k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$R_1^0 k \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ 3rd column of } A_1^0$$

$$R_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = -m_1 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1/2 \\ l_1 s_1/2 \\ 0 \end{bmatrix} - m_2 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1 + l_2 c_{12}/2 \\ l_1 s_1 + l_2 s_{12}/2 \\ 0 \end{bmatrix}$$

$K \rightarrow$ known

$P \rightarrow$ known

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k$$

$$\boxed{\begin{matrix} q_k = \theta_1 \\ \& \\ q_k = \theta_2 \end{matrix}}$$

Control of manipulators

The general form of equations

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$q \rightarrow$ joint angles/distance

$M \rightarrow$ mass matrix

$C \rightarrow$ Coriolis / Centripetal acceleration term

$G \rightarrow$ gravity matrix

$\tau \rightarrow$ External torque/force

Look at simplest example : one joint

$M = m$ $\quad$ $C = c$ (damping) $\quad$ $G = kq$ (spring)

$$\boxed{m\ddot{q} + c\dot{q} + kq = \tau = 0}$$ (spring mass damper)

$\omega_n^2 = \frac{k}{m}$ (natural frequency)

$2\xi\omega_n = \frac{c}{m}$ (damping)

3 m, k, c

$$\boxed{\ddot{q} + 2\xi\omega_n\dot{q} + \omega_n^2 q = 0}$$

2 ω_n, ξ

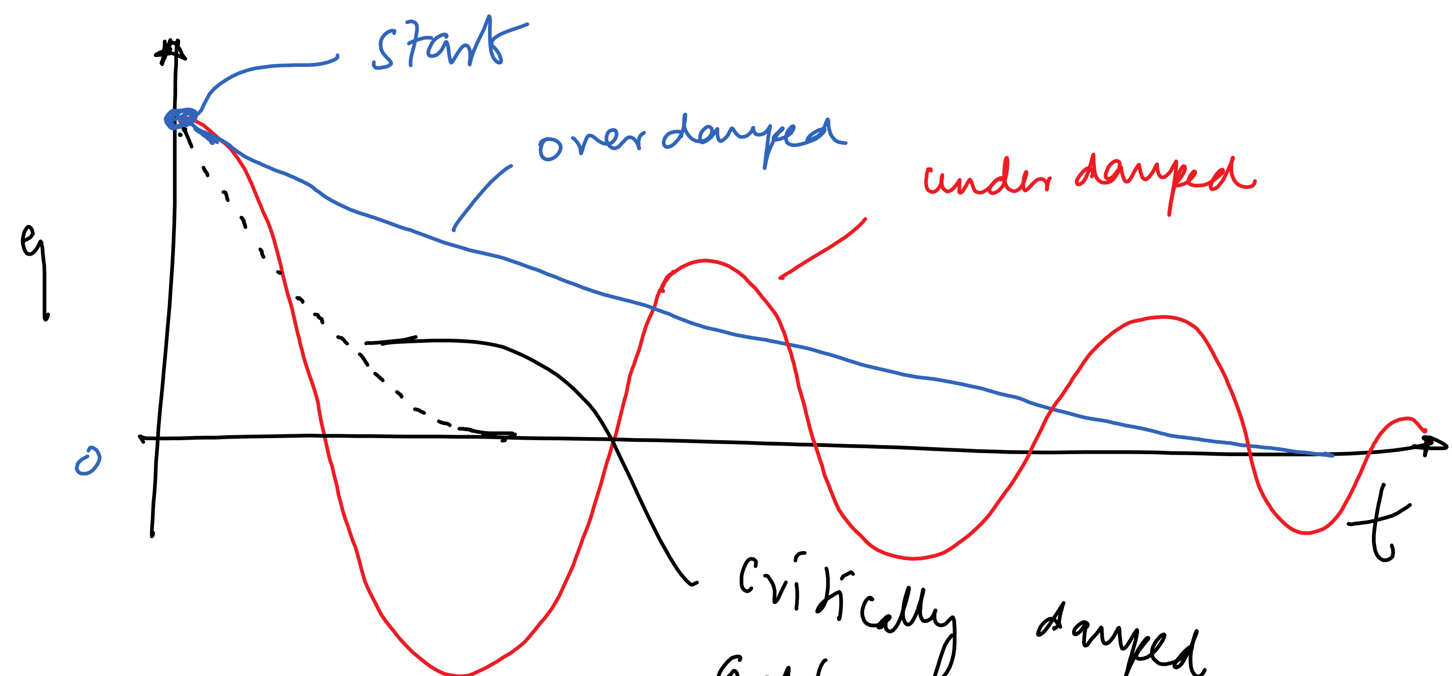
$2\xi\omega_n = c/m \Rightarrow 2\xi\sqrt{\frac{k}{m}} = c/m$

3 cases

① $\xi > 1 \Rightarrow c > 2\sqrt{km}$ overdamped

② $\xi = 1 \Rightarrow c = 2\sqrt{km}$ critical damped

③ $\xi < 1 \Rightarrow c < 2\sqrt{km}$ under damped.



critically damped achieve response $q \approx 0$ is shortest time

$\ddot{q} + 2\xi\omega_n\dot{q} + \omega_n^2 q = 0$ may not be critically damped but we can use τ to achive critical damping

$$m\ddot{q} + c\dot{q} + kq = \underline{\underline{\tau}}$$

Lets assume $\tau = -k_p q - k_d \dot{q}$

$k_p, k_d \rightarrow$ constants

p-d control

Substituting τ in the equation of motion

$$m\ddot{q} + c\dot{q} + kq = -k_p q - k_d \dot{q}$$

$$\Rightarrow \boxed{m\ddot{q} + (c + k_d)\dot{q} + (k + k_p) q = 0}$$

$$\omega_n^2 = \frac{k + k_p}{m} \qquad \xi = \frac{c + k_d}{2\sqrt{m(k + k_p)}}$$

Since ω_n, ξ are functions of k_p & k_d, we can set ω_n, ξ to any value we want.

Critical damping $\zeta = 1$

$$\zeta = \frac{c + k_d}{2\sqrt{m(k+k_p)}} = 1$$

$$c + k_d = 2\sqrt{m(k+k_p)}$$

Squaring both sides

$$(c + k_d)^2 = 4(m(k+k_p))$$

Simplify

$$k_d^2 + 2c\, k_d + \underbrace{c^2 - 4mk - 4mk_p}_{} = 0$$

Solution

$$k_d = \frac{-2c \pm \sqrt{(2c)^2 - 4(1)(c^2 - 4m(k+k_p))}}{2(1)}$$

ignore -ve term because that will make $k_d < 0$

$$\boxed{k_d = -c + 2\sqrt{m(k+k_p)}} \quad \text{— I}$$

Fix k_p & use I to find k_d

$m=1$, $c=1$, $k=10$ see MATLAB