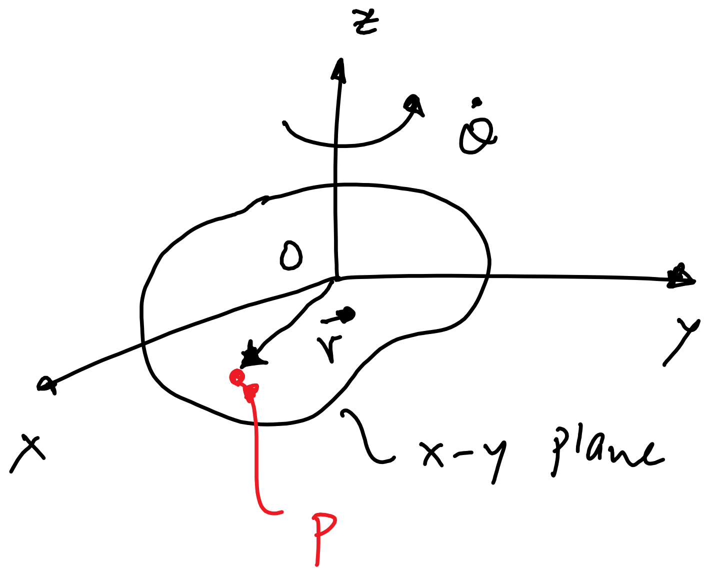
Angular / Linear velocity

Planar case

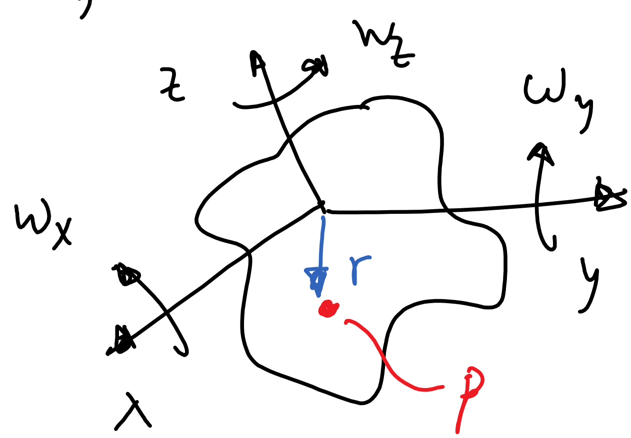


$\vec{r}$ — vector from O to P

$\vec{\omega}$ — angular velocity ; $\vec{\omega} = \dot{\theta}\hat{k}$ ✓

$\vec{V}_p$ ⟶ velocity of P ; $\vec{V}_p = \vec{\omega} \times \vec{r}$ ✓

3 D motion



X $\vec{\omega} \neq \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$

✓ $\vec{V}_p = \vec{\omega} \times \vec{r}$

↑ is true

Skew symmetric matrix

consider the vector $a = [a_x \ a_y \ a_z]$

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Properties of skew-symmetric matrix

(1) $S(a) + S^T(a) = 0_{3\times3}$

(2) $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$

α, β - constants

a, b - vectors

(3) $\vec{a} \times \vec{b} = \underset{3\times3}{S(a)} \underset{3\times1}{b}$ — 1st class

(4) $\boxed{S(Ra) = R S(a) R^T}$ R = rotation

(5) $X^T S X = 0$ $\underset{3\times1}{X}$ is any vector

Derivative of a rotation matrix

Consider $R(\theta)$ (rotation matrix)

$$R(\theta)\,R^T(\theta) = I$$ [property of a rotation matrix]

Differentiate with respect to θ

$$\frac{dR(\theta)}{d\theta} R^T(\theta) + \left(\left(R(\theta)\frac{dR^T(\theta)}{d\theta}\right)^T\right)^T = 0$$

$$\frac{dR(\theta)}{d\theta} R^T(\theta) + \left(\left(\frac{dR(\theta)}{d\theta}\right) R^T(\theta)\right)^T = 0$$

$(AB)^T = B^T A^T$

True

$$\underbrace{\left[\left(\frac{dR}{d\theta}\right) R^T\right]}_{S} + \underbrace{\left[\left(\frac{dR}{d\theta}\right) R^T\right]^T}_{S^T} = 0$$

$$S + S^T = 0$$

is a skew symmetric matrix

$$S = \left(\frac{dR}{d\theta}\right) R^T$$

$$\boxed{\underline{S} = \frac{dR(\theta)}{d\theta} R^T(\theta)}$$

Post multiply by $R(\theta)$

$$S \underline{R(\theta)} = \frac{dR(\theta)}{d\theta} \underbrace{R^T(\theta) R(\theta)}_{= I}$$

$$\boxed{\frac{dR(\theta)}{d\theta} = S R(\theta)}$$

<u>Example</u> For $R = R_{x,\theta}$

(1) Find $\frac{dR}{d\theta}$ and (2) S

(1) $R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ ✓

$$\boxed{\frac{dR_{x,\theta}}{d\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}}$$

(2) $S = ?$ $S = \frac{dR}{d\theta} R^T$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & +\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(\hat{i})$$

$$\frac{dR_{x,\alpha}}{d\alpha} = S(\hat{i})\, R_{x,\alpha}$$

Similarly

$$\frac{dR_{y,\alpha}}{d\alpha} = S(\hat{j})\, R_{y,\alpha}$$

$$\frac{dR_{z,\alpha}}{d\alpha} = S(\hat{k})\, R_{z,\alpha}$$

Angular velocity

Let $R = R(t)$ $\quad t \to$ time.

Assume : $\boxed{\dot{R}(t) = \frac{dR(t)}{dt} = S(\omega(t))\,R(t)}$ $\quad$ I

We know that

$$p^0 = R_1^0\, p^1$$

Diff. with respect to time

$$\dot{p}^0 = \frac{dp^0}{dt} = \frac{dR_1^0}{dt}\, p^1 = \dot{R}_1^0\, p^1 \quad \text{II}$$

Put I in II

$$\dot{p}^0 = S(\omega)\, R_1^0\, p^1 \quad = p^0$$

$$= S(\omega)\, p^0$$

$$\boxed{\dot{p} = \omega \times p^0}$$

$a \times b = S(a)\, b$

$\vec{V} = \omega \times \vec{r}$

$$\Rightarrow \boxed{\begin{aligned} \dot{R} &= S(\omega)\, R \\ S &= \dot{R}\, R^T \end{aligned}}$$

true

Notation

$$\underline{\underline{\dot{R}}} = S(\omega)\underline{\underline{R}}$$

$\omega^{k}_{i,j} \longrightarrow$ angular velocity associated with R^{i}_{j} is in the frame k

$$i = k = 0 \Rightarrow \omega^{0}_{0,j} = \omega_j$$

EXAMPLE

$$R = R_{x,\theta} \qquad \underline{\theta} = \underline{\omega}\,\underline{t}$$

Find $\dfrac{dR}{dt}$

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we know that $\dfrac{dR}{d\theta} = S(i)R$

$$\frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = \left(S(i)R\right)\omega$$

$$= S(\omega i)R = S(\omega_x)R$$

$$\boxed{\frac{dR_{x,\theta}}{dt} = S(\omega_x)R_{x,\theta}}$$

Addition of angular velocities

(or how to add angular velocities that involve different axis)

Consider the following rotation sequence

$$o_0 x_0 y_0 z_0 \longrightarrow o_1 x_1 y_1 z_1 \longrightarrow o_2 x_2 y_2 z_2$$

We known that ? unknown

$$\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0 \quad \text{—— (II)}$$

Alternate expression for $\dot{R}_2^0$

$$R_2^0 = R_1^0 R_2^1$$

Diff. w.r.t time

$$\dot{R}_2^0 = \underbrace{\dot{R}_1^0 R_2^1}_{(1)} + \underbrace{R_1^0 \dot{R}_2^1}_{(2)} \quad \text{—— (I)}$$

(1) $\dot{R}_1^0 R_2^1 = [S(\omega_{0,1}^0) R_1^0] R_2^1 = S(\omega_{0,1}^0) R_2^0$

② $R_1^0 \dot{R}_2^1 = R_1^0 \left[S(\omega_{1,2}^1) R_2^1 \right]$

$$= R_1^0 \, S(\omega_{1,2}^1) \underbrace{\left[(R_1^0)^T R_1^0 \right]}_{= I} R_2^1$$

$$= R_1^0 \, S(\underline{\omega_{1,2}^1}) \, (R_1^0)^T \, R_2^0$$

Recall $S(R\omega) = R \, S(\omega) \, R^T$

$$= \underline{S(R_1^0 \omega_{1,2}^1) \, R_2^0}$$

$$\dot{R}_2^0 = S(\omega_{0,1}^0) R_2^0 + S(R_1^0 \omega_{1,2}^1) R_2^0$$

From I ① ②

$$\dot{R}_2^0 = \left[S(\omega_{0,1}^0) + S(R_1^0 \omega_{1,2}^1) \right] R_2^0$$

$$\dot{R}_2^0 = S(\omega_{0,1}^0 + R_1^0 \omega_{1,2}^1) \, R_2^0 \quad \text{— III}$$

But $\dot{R}_2^0 = S(\omega_{0,2}^0) \, R_2^0$ From II

From II & III

$$\boxed{\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \, \omega_{1,2}}$$

Generalize to n frames

$$\boxed{\omega^0_{0,n} = \omega^0_{0,1} + R^0_1 \, \omega^1_{1,2} + R^0_2 \, \omega^2_{2,3} + R^0_3 \, \omega^3_{3,4} + \dots + R^0_{n-1} \, \omega^{n-1}_{n-1,n}}$$

$$\omega^0_{0,n} = \omega^0_{0,1} + \omega^0_{1,2} + \omega^0_{2,3} + \dots \omega^0_{n-1,n}$$

where $\omega^0_{1,2} = R^0_1 \, \omega^1_{1,2}$

$\omega^0_{2,3} = R^0_2 \, \omega^2_{2,3}$

$\vdots$

$\omega^0_{n-1,n} = R^0_{n-1} \, \omega^{n-1}_{n-1,n}$