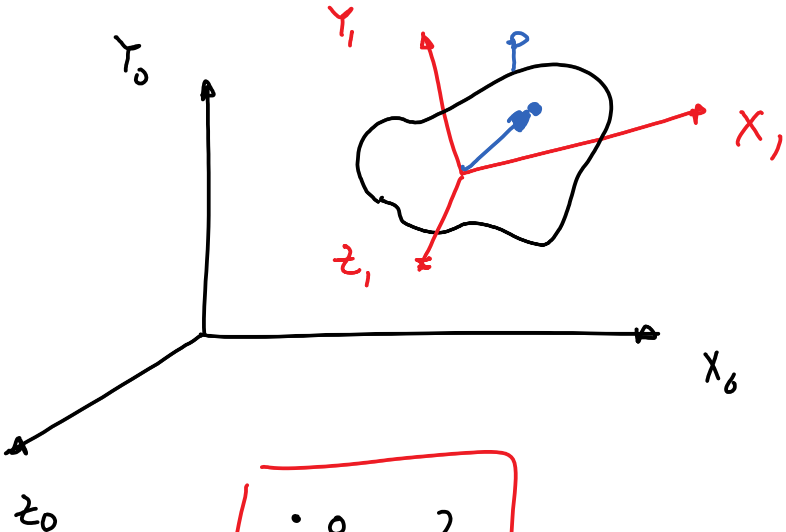
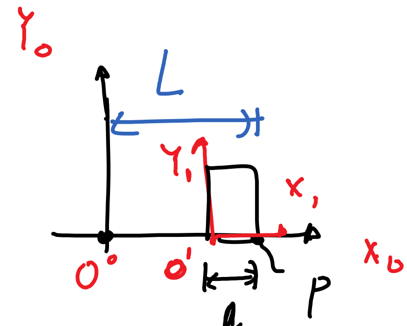
Linear velocity of a point attached to a moving frame



$$\boxed{\dot{p}^0 = ?}$$



$p^1 = l$

$p^0 = L$

$$\Rightarrow \underline{p}^0 = \underline{\underline{H}}_1^0 \, \underline{p}^1$$

$$\therefore \begin{bmatrix} p^0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p^1 \\ 1 \end{pmatrix}$$

$$p^0 = \underline{R_1^0} \, p^1 + o_1^0$$

$$\dot{p}^0 = \dot{R}_1^0 \, \underline{\underline{p^1}} + R_1^0 \dot{p}^1 + \dot{o}_1^0$$

Aside

$$p^0 - o_1^0 = R_1^0 p$$

$$(R_1^0)^T (p^0 - o_1^0) = R_1^{0T} R_1^0 p^1$$

$$(R_1^0)^T (p^0 - o_1^0) = \underline{p^1}$$

$$\dot{p}^0 = \overbrace{(\dot{R}_1^0)(R_1^0)^T}^{S(\omega)} (p^0 - o_1^0) + R_1^0 \dot{p}^1 + \dot{o}_1^0$$

$$\boxed{\dot{p}^0 = \omega \times (p^0 - o_1^0) + R_1^0 \dot{p}^1 + \dot{o}_1^0}$$ [Using $a \times b = S(a) b$]

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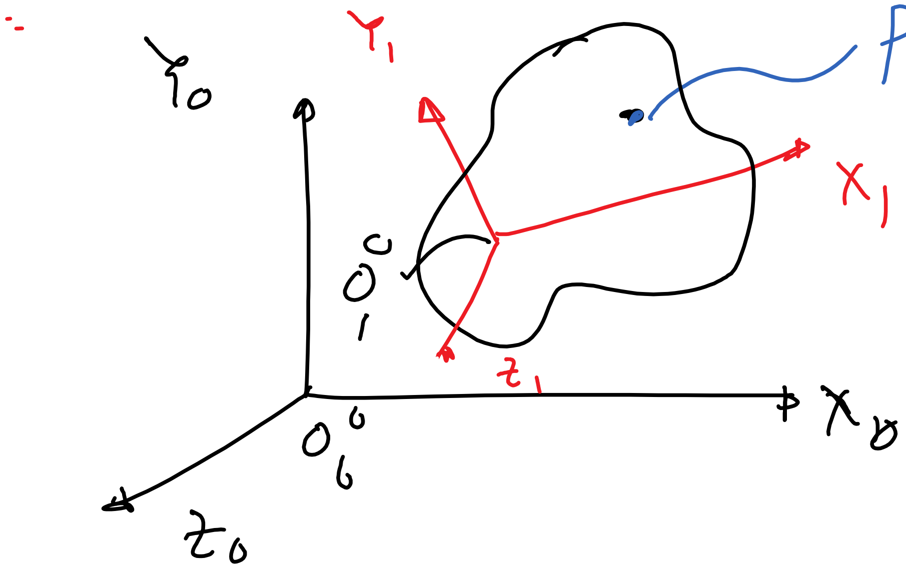


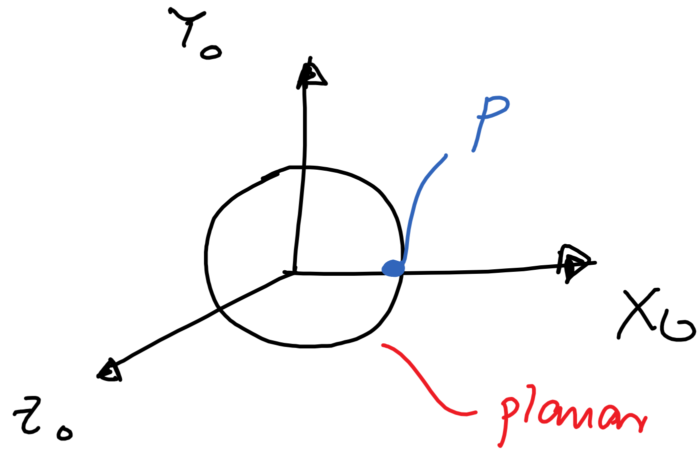
$\dot{p}^0$ — How p moves w.r.t. frame $O_0 X_0 Y_0 Z_0$ (global/fixed frame)

p^0 — position in global frame

$$\boxed{\dot{p}^1 \text{ — How } p \text{ moves wrt. frame } X_1 Y_1 Z}$$

$\dot{O}_1^0$ — Origin of frame $X_1 Y_1 Z_1$ moves with time

EXAMPLE



planar disc in the plane X_0Y_0

about z_0 axis

The disc is spinning with constant angular velocity of ω. At time $t=0$, the point P is at a distance 'a' along the x-axis. Find velocity of point P at time t

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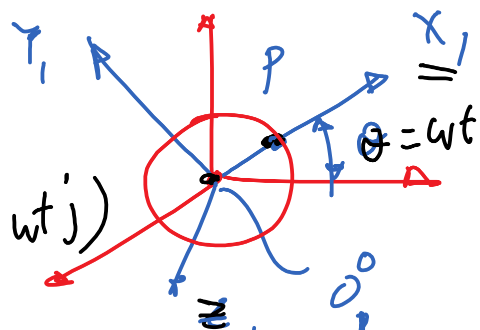
$$\dot{p}^0 = \omega \times (p^0 - \cancel{o_1^0}) + \cancel{R_1^0 \dot{p}^1} + \cancel{\dot{o}_1^0}$$

$$\boxed{\dot{o}_1^0 = 0 \; ; \; \dot{p}^1 = 0 \; ; \; o_1^0 = 0}$$

$$\dot{p}^0 = \omega \times p^0$$

$$= \omega \hat{k} \times (a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j})$$



$$\boxed{p^0 = R_1^0 p^1 = \begin{bmatrix} \cos(\omega t) & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} a\cos\omega t \\ a\sin\omega t \\ 0 \end{bmatrix}}$$

$$p^0 = R_1^0\, p^1 = \begin{bmatrix} \cos(\omega t) & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} a\cos\omega t \\ a\sin\omega t \\ 0 \end{bmatrix}}$$

$$\dot{p} = \omega k \times \{ a \cos \omega t\, \hat{i} + a \sin \omega t\, \hat{j} \}$$

$$\boxed{\dot{p}^0 = \begin{bmatrix} -\omega a \sin(\omega t) \\ \omega a \cos(\omega t) \\ 0 \end{bmatrix}}$$

<u>Another way</u> $\dot{p} = \underline{\underline{\omega}} \times p^0$

$$\dot{p} = S(\omega)\, p^0$$

$$\dot{p}^0 = (\dot{R}_1^0)^T (R_1^0)\, p^0$$

$$\dot{R}_1^0 = \frac{d}{dt}\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(\omega t)\,\omega & -\cos(\omega t)\,\omega & 0 \\ \cos(\omega t)\,\omega & -\sin(\omega t)\,\omega & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{p} = (\dot{R}_1^0)^T R_1^0\, p^0 \quad \checkmark \quad \checkmark \quad \checkmark$$

$$\begin{bmatrix} a \cos \omega t \\ a \sin \omega t \\ 0 \end{bmatrix}$$

Jacobian:

Jacobians may be used to find
(a) velocity of points / bodies
(b) static forces.

The position/orientation of the end-effector is given by A_n^0

$$A_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$$

To find velocities / angular velocity of the end-effector link.

$$\left.\begin{aligned} V_n^0 &= \dot{o}_n^0 \\ S(\omega_n^0) &= \dot{R}_n^0 \, (R_n^0)^T \end{aligned}\right\} \text{ I}$$

Probably the hard way

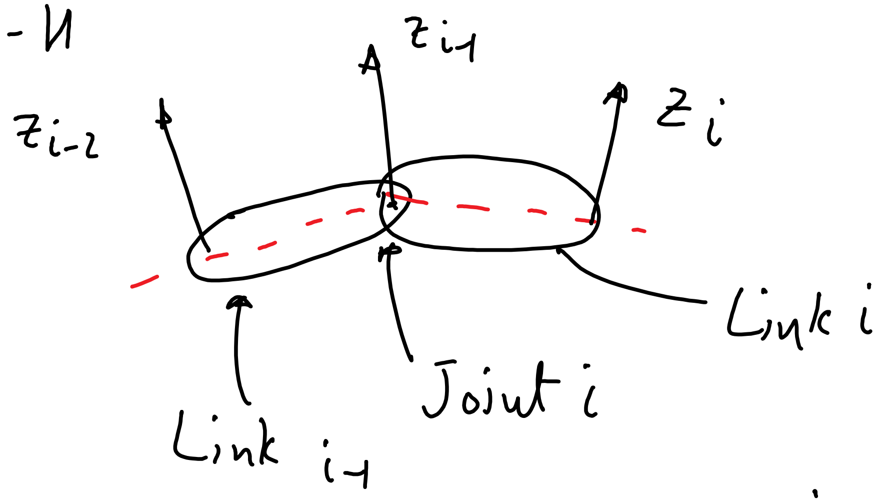
Easier way & more compact

$$\left.\begin{aligned} V_n^0 &= J_v \dot{q} \\ \omega_n^0 &= J_\omega \dot{q} \end{aligned}\right\} \rightarrow \underbrace{\begin{bmatrix} V_n^0 \\ \omega_n^0 \end{bmatrix}}_{6\times 1} = \underbrace{\begin{bmatrix} J_V \\ J_\omega \end{bmatrix}}_{6\times n} \underbrace{\dot{q}}_{n\times 1}$$

Jacobian

Computing J_ω

D-H



i^{th} joint is revolute : $\omega_{i-1,i}^{i-1} = \dot{\theta}_i \hat{k}$

" " is prismatic : $\omega_{i-1,i}^{i-1} = 0$

For the end-effector (derived in the last class)

$$\omega_n^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \dots R_{n-1}^0 \omega_{n-1,n}^0$$

$$= \rho_1 \dot{q}_1 \hat{k} + R_1^0 \rho_2 \dot{q}_2 \hat{k} + R_2^0 \rho_3 \dot{q}_3 \hat{k} + \dots R_{n-1}^0 \rho_n \dot{q}_n \hat{k}$$

$\Rightarrow$ $\rho_i = 0$ (prismatic) $\rho_i = 1$ (revolute)

$$\omega_n^0 = \begin{bmatrix} \rho_1 k & \rho_2 R_1^0 k & \rho_3 R_2^0 \hat{k} & \dots & \rho_n R_n^0 \hat{k} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_n \end{bmatrix}$$

$$\omega_n^0 = J_\omega \dot{q}$$

$$\begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Computing J_v

$$v_n^0 = \dot{O}_n^0 = \sum_{i=1}^{n} \frac{\partial O_n}{\partial q_i} \dot{q}_i \quad \text{(By chain rule)}$$

$$= \underbrace{\left[\frac{\partial O_n}{\partial q_1} \quad \frac{\partial O_n}{\partial q_2} \quad \cdots \quad \frac{\partial O_n}{\partial q_n} \right]}_{J_v} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$v_n^0 = J_v \dot{q}$$

We will derive a formula for J_v