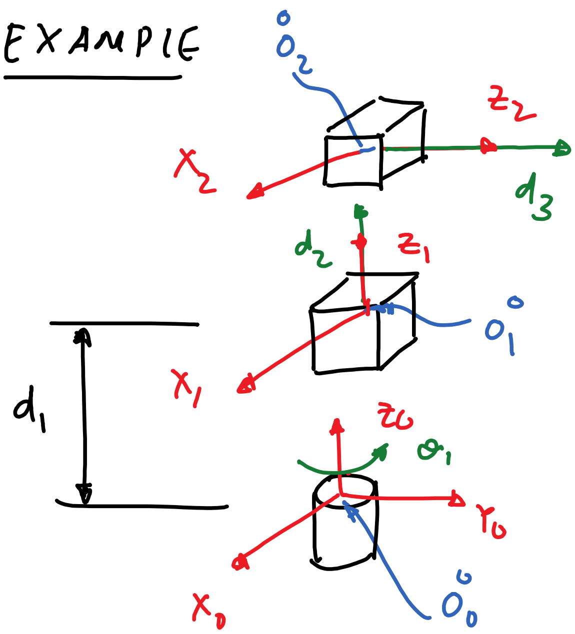
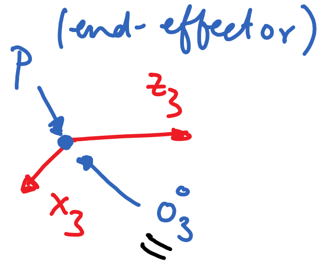
EXAMPLE





Link i	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1
2	0	$-\pi/2$	d_2	0
3	0	0	d_3	0

Find the linear and angular velocity of the point P (the end-effector) for the three link manipulator shown above

$$\underset{6\times 1}{\begin{bmatrix} V_P \\ \omega_P \end{bmatrix}} = \underset{6\times 3}{\begin{bmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{\omega_1} & J_{\omega_2} & J_{\omega_3} \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} \dot\theta_1 \\ \dot d_2 \\ \dot d_3 \end{bmatrix}} = \begin{bmatrix} R_0^0 k \times (O_3^0 - O_0^0) & R_1^0 k & R_2^0 k \\ R_0^0 k & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot d_2 \\ \dot d_3 \end{bmatrix} \quad \text{— I}$$

? ?

Recall:

$$J_{v_i} = \begin{cases} R_{i-1}^0 \, k \times (O_n - O_{i-1}^0). & \text{Revolute} \\ R_{i-1}^0 \, k & \text{Prismatic} \end{cases}$$

$$J_{\omega_i} = \begin{cases} R_{i-1}^0 \, K & \text{Revolute} \\ 0_{1\times 3} & \text{Prismatic} \end{cases}$$

$$c_1 = \cos\theta_1 \qquad s_1 = \sin\theta_1$$

R^0_0

$$A^0_0 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R^0_1 =$

$$A^0_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^1_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^2_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R^0_2 =$

$$A^0_2 = A^0_1 A^1_2 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$O^0_3 =$

$$A^0_3 = A^0_2 A^2_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -d_3 s_1 \\ s_1 & 0 & c_1 & d_3 c_1 \\ 0 & -1 & 0 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$O^0_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

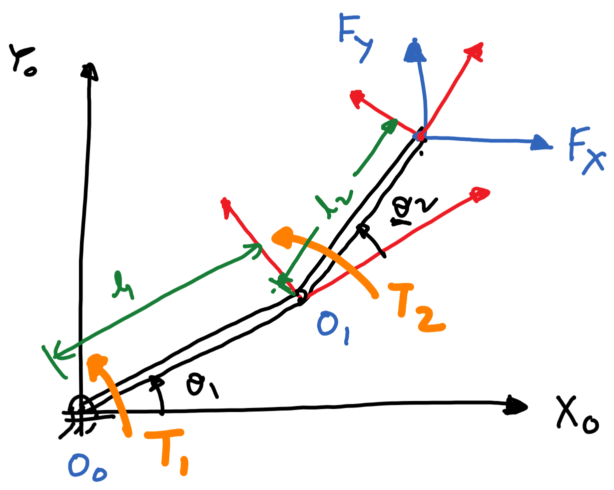
$$R^0_0 k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad R^0_1 k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad R^0_2 \hat{k} = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}$$

Put in (I)

$$\begin{bmatrix} v_p \\ w_p \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{pmatrix} -d_3 c_1 \\ -d_3 s_1 \\ 0 \end{pmatrix}}^{J_{v_1}} & \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{J_{v_2}} & \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{J_{v_3}} \\ \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{J_{\omega_1}} & \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{J_{\omega_2}} & \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{J_{\omega_3}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$\left(\omega_p\right) \quad \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} J_{\omega_1} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} J_{\omega_2} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} J_{\omega_3} \right] \left(\dot{d}_3 \right)$$

EXAMPLE



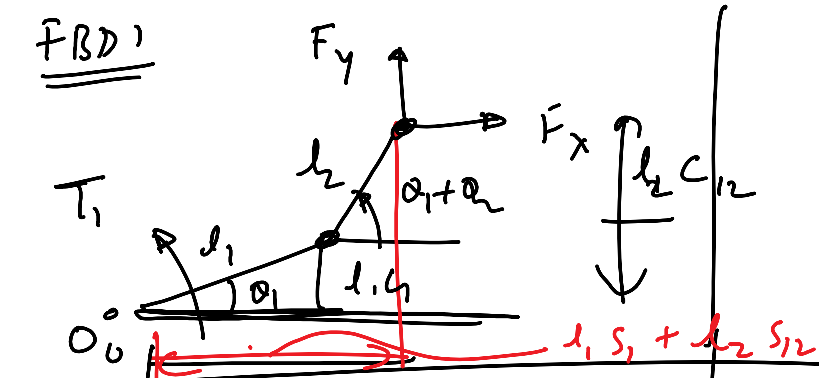
D-H Table

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2

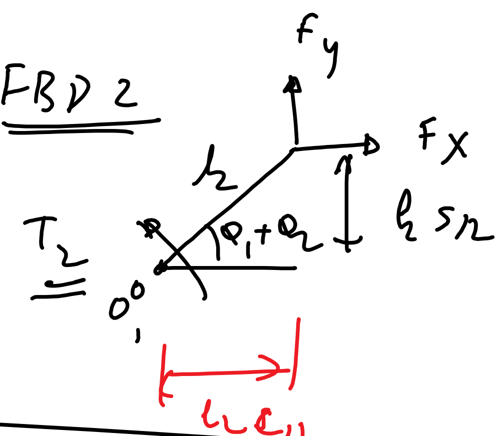
Consider the two link planar manipulator shown above. There is a force on the end-effector F = [Fx, Fy,0] as shown. Find the torques T1 and T2 needed to keep the manipulator in static equilibrium. Assume that the manipulator is in the horizontal place so gravity may be ignored.

Method 1 : Free Body diagram

FBD 1



FBD 2



Equation 1 Take moment about O_0^0

$$+T_1 = -F_x\{l_1 c_1 + l_2 c_{12}\} + F_y\{l_1 s_1 + l_2 s_{12}\}$$

$c_1 = \cos\theta_1 \quad s_1 = \sin\theta_1$

$c_{12} = \cos(\theta_1+\theta_2) \quad s_{12} = \sin(\theta_1+\theta_2)$

$$\boxed{\Sigma T_{ext} = \Sigma T_{int}}$$ — correction

Equation 2 Take moments about O_1^0

$$T_2 = -F_x l_2 s_{12} + F_y l_2 c_{12}$$

Solve for T_1 & T_2

$$\boxed{T_2 = -F_x l_2 s_{12} + F_y l_2 c_{12}}$$

$$\boxed{\sum T_{ext} = \sum T_{int}}$$ - correction

Formula for Torque using Jacobians

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \end{bmatrix}$$

$$\dot{x} = J \dot{q}$$

$$\frac{dx}{dt} = J \frac{dq}{dt} \Rightarrow \delta x = J \delta q \quad \text{--- (1)}$$

If F is the force at δx and T is the torque at δq

$$\delta W = F^T \delta x - T^T \delta q = 0$$

principle of virtual work

$$F^T \delta x = T^T \delta q \quad \text{--- (2)}$$

Put (1) in (2) $\quad F^T J \delta q = T^T \delta q$

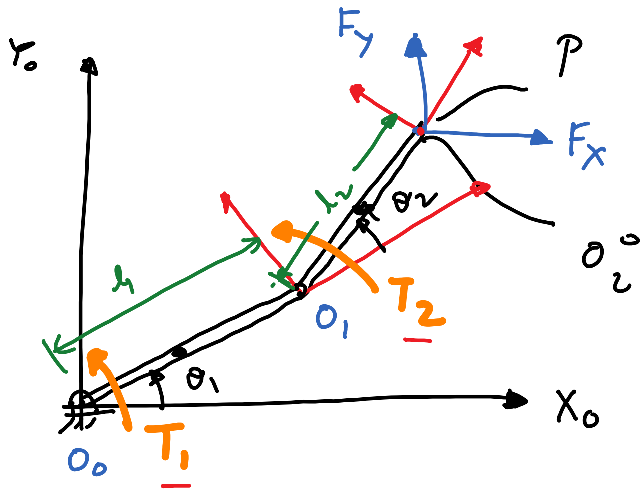
$\delta q \neq 0 \qquad F^T J = T^T$

Take transpose of both side $\quad (F^T J)^T = (T^T)^T$

$$\boxed{J^T F = T}$$

$$(AB)^T = B^T A^T$$

EXAMPLE



D-H Table

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2	l_2	0	0	θ_2

Consider the two link planar manipulator shown above. There is a force on the end-effector F = [Fx, Fy,0] as shown. Find the torques T1 and T2 needed to keep the manipulator in static equilibrium. Assume that the manipulator is in the horizontal place so gravity may be ignored.

Method 2 : Using Jacobian

$$T = J_p^T F_p$$

$$F_p = \begin{bmatrix} \underbrace{\begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix}}_{\text{forces in } x,y,z} \\ \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{Torques at } p \text{ in } x,y,z} \end{bmatrix}$$

$$J_p = \begin{bmatrix} J_{v_1} & J_{v_2} \\ J_{w_1} & J_{w_2} \end{bmatrix}$$

$$= \begin{bmatrix} R_0^0 k \times (o_2^0 - o_0^0) & R_1^0 k \times (o_2^0 - o_1^0) \\ R_0^0 k & R_1^0 k \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{\begin{pmatrix} -l_1 S_1 - l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} \\ 0 \end{pmatrix}}^{J_{v_1}} & \overbrace{\begin{pmatrix} -l_2 S_{12} \\ l_2 C_{12} \\ 0 \end{pmatrix}}^{J_{v_2}} \\ \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= J_{w_1}} & \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{J_{w_2}} \end{bmatrix}$$

6×2

$$T = J^T F$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & l_1 c_1 + l_2 c_{12} & 0 & 0 & 0 & 1 \\ -l_2 s_{12} & +l_2 c_{12} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2×1 $\quad$ 2×6 $\quad$ 6×1

$$\boxed{\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -F_x (l_1 s_1 + l_2 s_{12}) + F_y (l_1 c_1 + l_2 c_{12}) \\ -F_x (l_2 s_{12}) + F_y l_2 c_{12} \end{bmatrix}}$$

$$T_2 = -F_x l_2 s_{12} + F_y l_2 c_{12}$$

— From Method 1

~~The sign is incorrect. I will check this.~~

Fixed. Method 1 had an error.

Jacobians and singularity

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

find   given

$$\dot{q} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}^{-1} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

find   given

what if

$$\det\left(\begin{pmatrix} J_v \\ J_\omega \end{pmatrix}^{-1}\right) = 0$$

$$\underbrace{\frac{1}{\det}}_{=0}\left(\text{cofactor}\begin{pmatrix} J_v \\ J_w \end{pmatrix}\right)$$

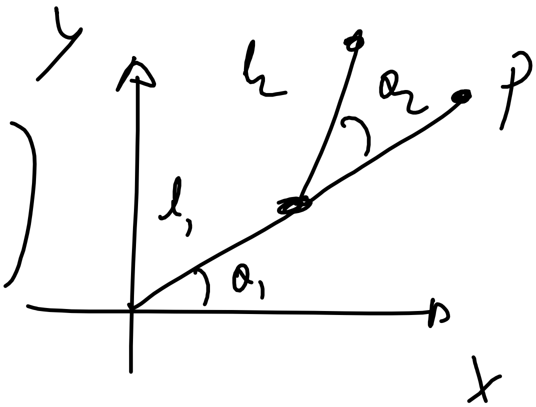
the inverse is going to be ∞.

$$\dot{q} \rightarrow \infty$$

issue.

Example of singularity

$$\begin{bmatrix} V_{px} \\ V_{py} \end{bmatrix} = \underbrace{\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}}_{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



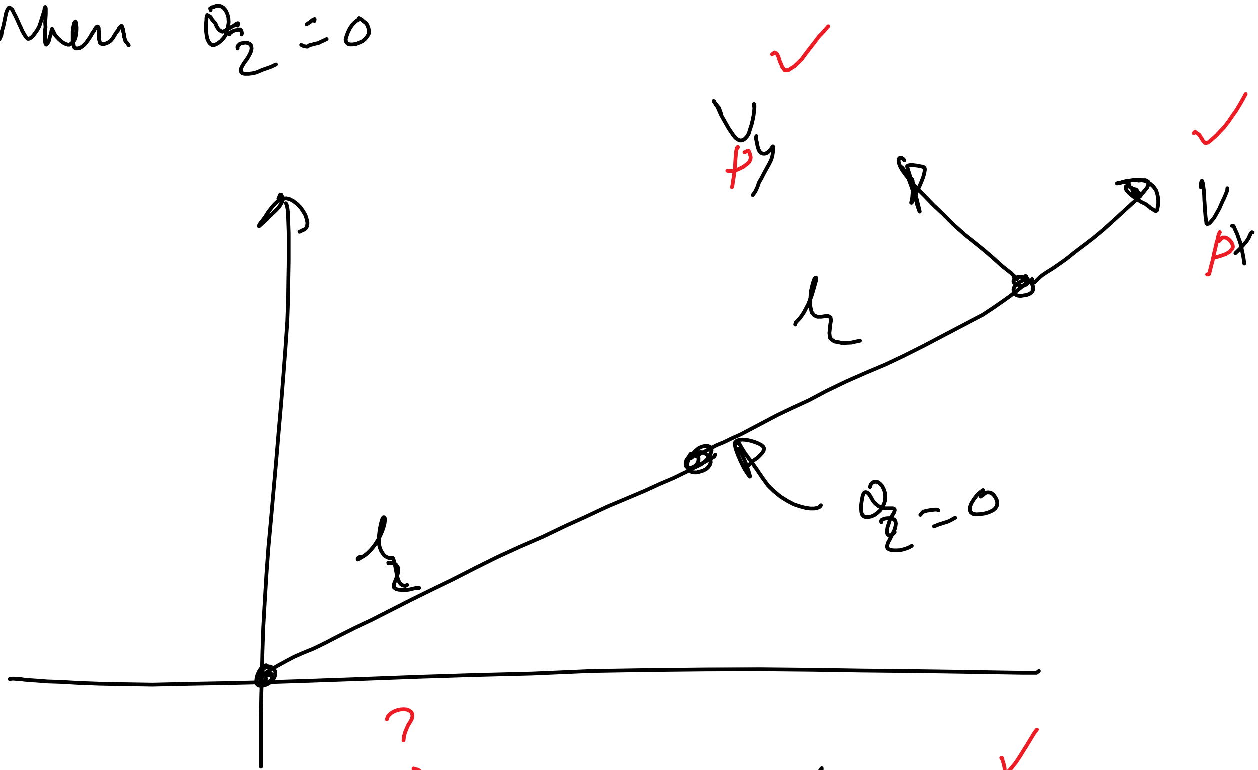
Lets put $l_1 = l_2 = 1$ and then take the inverse of the Jacobian (J)

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta_1+\theta_2)/\sin\theta_2 & \sin(\theta_1+\theta_2)/\sin\theta_2 \\ -\cos((\theta_1+\theta_2)/2)/\sin(\theta_2/2) & -\sin((\theta_1+\theta_2)/2)/\sin(\theta_2/2) \end{bmatrix}} \begin{bmatrix} V_{px} \\ V_{py} \end{bmatrix}$$

Becomes singular when $\theta_2 = 0$

$J^{-1} \rightarrow \infty$

When $\theta_2 = 0$



$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = J^{-1} \begin{pmatrix} v_{px} \\ v_{py} \end{pmatrix}$$

? $\quad J^{-1} \to \infty$

Here the manipulator cannot move in the V_{px} direction but can only move in the V_{py} direction. This is the singular position because one cannot find $\dot{\theta}_1$ & $\dot{\theta}_2$ to achive a particular $\underline{\underline{V_{px}}}$.