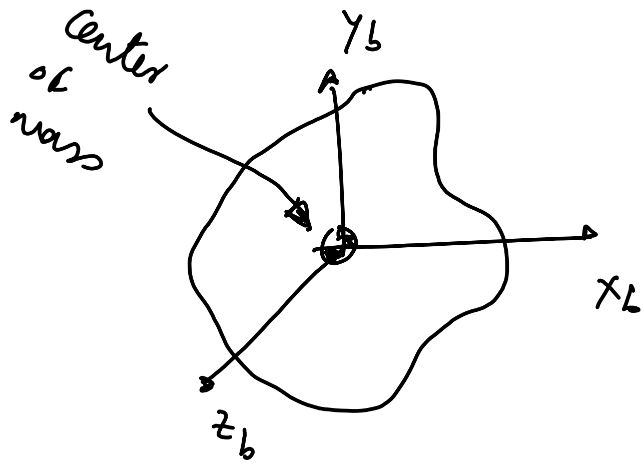
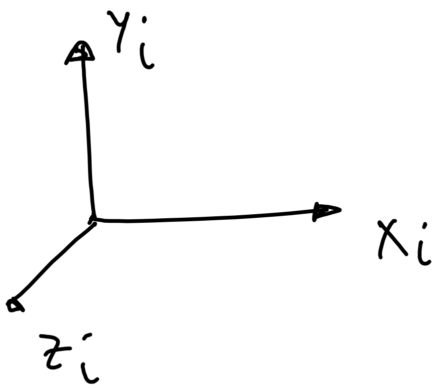
Equations of motion for a manipulator

kinetic energy





i — world / fixed / global frame

b — body frame
(rotates & translates as the body moves)

$i = 0$ in our discussion so far on manipulators

$$\boxed{K = \frac{1}{2} m v_i^T v_i + \frac{1}{2} \omega_i^T I_i \omega_i}$$

$$v_i = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$\underbrace{v_i, \omega_i}$ are linear and angular velocities in world / global frame

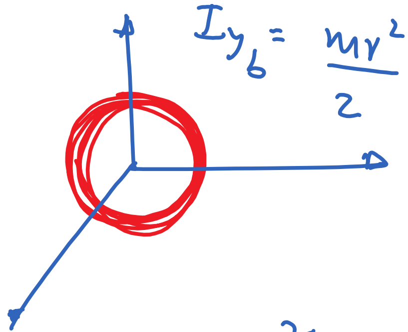
$$\omega_i = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

we have derived formula for v_i, ω_i { = recap ω_0, v_0 }

However, I_i is not readily available only I_b is available.

I_b is easily derived. It is also available in dynamics text as Inertia tables

e.g Ring



$i = 0$ — o world frame

However, we need I_i. R_b^i

If we know the $\underbrace{\text{rotation matrix}}_{R_b^i}$ that relates x_b, y_b, z_b with x_i, y_i, z_i then it is easy to relate I_b and I_i

$$I_i = R_b^i \, I_b \, (R_b^i)^T$$

Thus

$$K = \frac{1}{2} m v_i^T v_i + \frac{1}{2} \omega_i^T R_b^i I_b (R_b)^T \omega_i$$

$$\boxed{K = \sum_{i=1}^{n} \left(\frac{1}{2} m v_i^T v_i + \frac{1}{2} w_i^T R_b^i I_b R_b^T w_i \right)}$$

$S(w) = \dot{R} R^T$ (complex)

$v_i = J_{v_i} \dot{q}$ and $\omega_i = J_{\omega_i} \dot{q}$

sub in K expression

$$\boxed{K = \frac{1}{2} \dot{q}^T \underbrace{\left\{ \sum_{i=1}^{n} m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_b^i I_b R_b^{i\,T} J_{\omega_i} \right\}}_{D} \dot{q}}$$

$$K = \frac{1}{2} \dot{q}^T D \dot{q}$$

<u>Potential energy of the links</u>

$$P = -m_i g^T r_{c_i}$$

{ r_{c_i} is the center of mass of i^{th} link }

$$\boxed{P = -\sum_{i=1}^{n} m_i g^T r_{c_i}}$$

-ive sign because positive $r_{c_i} \uparrow$ in presence of $-g \downarrow$ should give positive potential

Note the change from video

Equations of motion

$$L = K - P$$

$$L = \frac{1}{2} \underline{\dot{q}^T D \dot{q}} + \sum_{i=1}^{n} \underline{m_i g^T r_{c_i}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial q_k} = \tau_k$$

$\dot{q}_k \rightarrow k^{th}$ element of $\dot{q}$

$$\frac{\partial L}{\partial \dot{q}_k} = D\dot{q}_k \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(D\dot{q}_k\right)$$

$$= \dot{D}\underline{\dot{q}_k} + D\underline{\ddot{q}_k}$$

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\dot{q}^T \frac{\partial D}{\partial q_k}\dot{q} + \sum_{i=1}^{n} m_i g^T \frac{\partial r_{c_i}}{\partial q_k}$$

Putting all terms together, we have

$$\underbrace{D}_{D(q)}\ddot{q}_k + \underbrace{\dot{D}\dot{q}_k + \frac{1}{2}\dot{q}^T\frac{\partial D}{\partial q_k}\dot{q}}_{c(q,\dot{q})\dot{q}} + \underbrace{\left(+\sum_{i=1}^{n} m_i g^T \frac{\partial r_{c_i}}{\partial q_k}\right)}_{g(q)} = \tau_k$$

$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

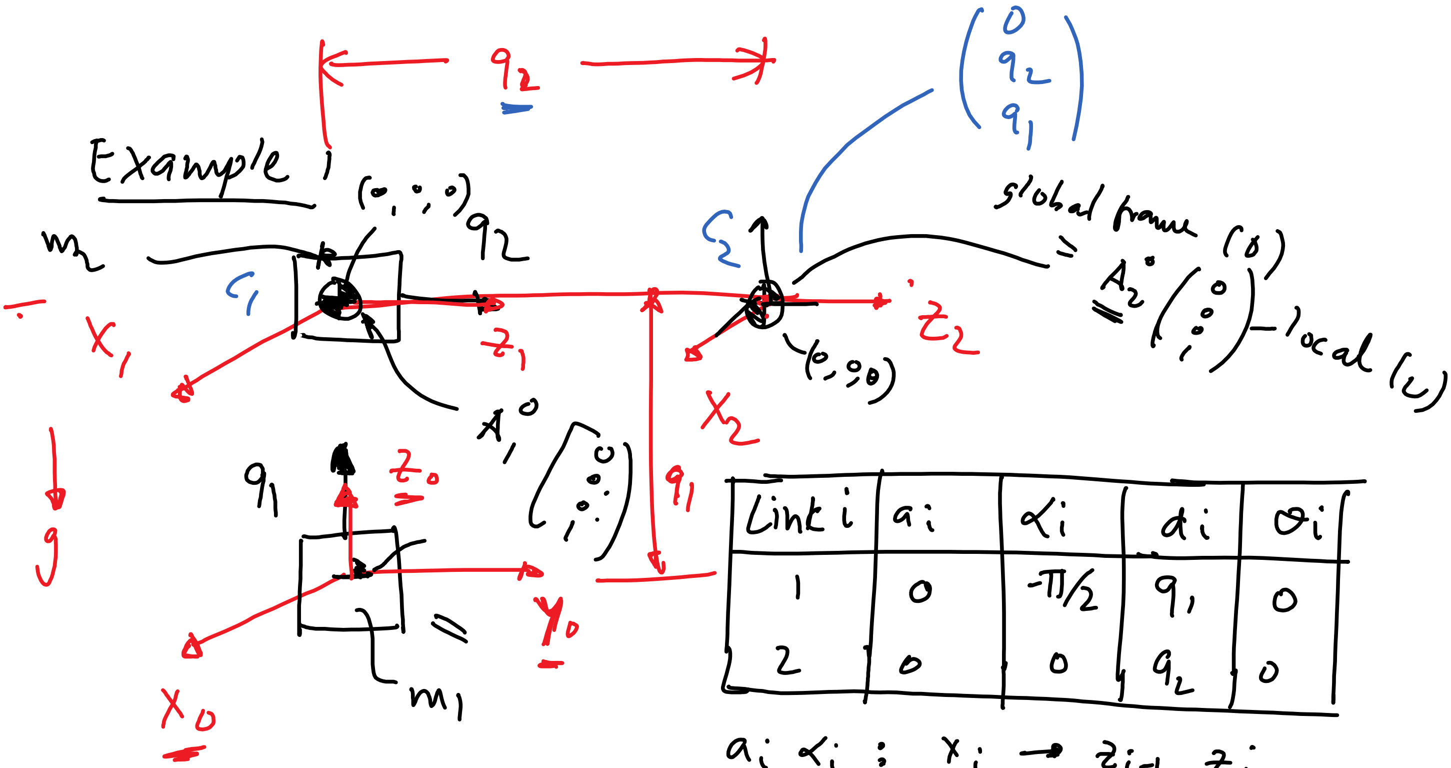
all n equations for all links

Equation of motion of a manipulator

Finding the equations of motion

1. Assign frames
2. Create D-H table
3. Find J_{v_i}, J_{ω_i}, r_{c_i}
4. Find $\mathcal{L} = K - P = \frac{1}{2}\dot{q}^T D \dot{q} + \sum_{i=1}^{n} m_i g^T r_{c_i}$
5. Use $\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} = \tau_k$

$k = 1, 2, 3, \ldots n$



Link i	a_i	α_i	d_i	θ_i
1	0	$-\pi/2$	q_1	0
2	0	0	q_2	0

a_i, α_i : $x_i \to z_{i-1}, z_i$

$d_i\ \theta_i$: $z_{i-1} \to x_{i-1}, x_i$

Question: For the 2 link manipulator shown above find the equations of motion.
Assume that the mass of the links are m1 and m2 as shown in the figure

① , ② done. We need to find ③ J_v, J_w, r_c

④ Find $\mathcal{L}$ ⑤ Find equations of motion from $\mathcal{L}$.

$$A_0^0 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (R_0^0 k)$$

$$A_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_1^0 k)$$

$$A_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^0 = A_1^0 A_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & q_2 \\ 0 & -1 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ Find $J_{w_1}, J_{v_1}, J_{w_2}, J_{v_2}, r_{c_1}, r_{c_2}$

For prismatic joint : $J_{v_i} = R^0_{i-1} k \quad J_{\omega_i} = 0$

$$J_{\omega_1} = \underset{3 \times 2}{0} \qquad J_{\omega_2} = 0_{3\times 2}$$

rows columns

$$J_{v_1} = [\underbrace{\underline{\underline{R^0_0}} k}_{q_1} \quad \underbrace{0}_{q_2}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

movement does not affect C_1

$$J_{v_2} = [R^0_0 k \quad R^0_1 k] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$r_{c_1} = A^0_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ q_1 \\ 1 \end{pmatrix}$$

$$r_{c_2} = A^0_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ q_2 \\ q_1 \\ 1 \end{pmatrix}$$

④ $K = \frac{1}{2} \dot{q}^T \{ m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + \ldots$

$$\underline{\underline{J_{w_1}}}^T R_b I_1 R_b^T J_{w_1} + \underline{\underline{J_{w_2}}} R_b I_2 R_b^T J_{w_2} \} \dot{q}$$

$$= \frac{1}{2} \dot{q}^T \left\{ m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \dot{q}$$

$$= \frac{1}{2} \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$K = \frac{1}{2} (m_1 + m_2) \dot{q}_1^2 + m_2 \dot{q}_2^2$$

$$P = - m_1 g^T r_{c_1} - m_2 g^T r_{c_2}$$

$$= - m_1 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ q_1 \end{bmatrix} - m_2 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} 0 \\ q_2 \\ q_1 \end{bmatrix}$$

$$P = + m_1 g q_1 + m_2 g q_1 = + (m_1 + m_2) g q_1$$

+ive potential is positive if ↑ movement tive potential is positive if T movement

$$\mathcal{L} = K - P$$

$$\mathcal{L} = \frac{1}{2}\underline{(m_1+m_2)}\dot{q}_1^2 + \underline{m_2}\dot{q}_2^2 + \underline{(m_1+m_2)}g\,q_1$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k$$

(i) $q_k = q_1$

$$\frac{d}{dt}\left((m_1+m_2)\dot{q}_1\right) + \left((m_1+m_2)g\right) = f_1$$

$$\boxed{(m_1+m_2)\ddot{q}_1 + (m_1+m_2)g = f_1}$$

(ii) $q_k = q_2$

$$\frac{d}{dt}\left(m_2\dot{q}_2\right) + 0 = f_2$$

$$\boxed{m_2\ddot{q}_2 = f_2}$$