Dynamics

Find equations of motion of a ball falling in gravity and subject to a quadratic drag force, $\vec{F} = -c|\vec{v}|^2\hat{v}$, where c is a constant, $\vec{v}$ is the velocity, $\hat{v}$ is a unit vector for the velocity.

NOTE:

$$\vec{F} = F_x\hat{i} + F_y\hat{j} = -c|\vec{v}|^2\hat{v}$$

$$= -c|\vec{v}|^{\cancel{2}}\frac{\vec{v}}{\cancel{|\vec{v}|}}$$

$$= -c|\vec{v}|\vec{v}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} = -c\sqrt{\dot{x}^2+\dot{y}^2}\{\dot{x}\hat{i}+\dot{y}\hat{j}\}$$

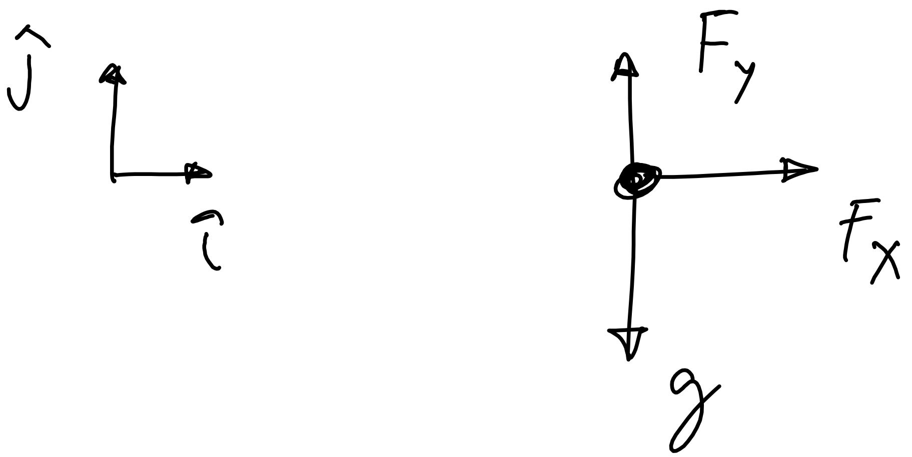
$\Rightarrow$ Comparing both sides

$$\rightarrow F_x = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x} \quad \text{— I}$$

$$F_y = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y} \quad \text{— II}$$

Solution

Free Body Diagram (FBD)



Newton's 2nd law $\vec{F}_{ext} = m\vec{a}$

$$= m\ddot{x}\,\hat{i} + m\ddot{y}\,\hat{j}$$

$$\vec{F}_{ext} = m\ddot{x}\,\hat{i} + m\ddot{y}\,\hat{j} = -mg\,\hat{j} + F_x\,\hat{i} + F_y\,\hat{j}$$

Compare $\hat{i}$ and $\hat{j}$ components.

$$\Rightarrow \quad m\ddot{x} = +F_x = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x} \quad \text{From } \mathrm{I}$$

$$\boxed{\ddot{x} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}} \quad \text{— III}$$

From II

$$\Rightarrow \quad m\ddot{y} = -mg + F_y = -mg - c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}$$

$$\boxed{\ddot{y} = -g - \frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}}$$

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Euler-lagrange method

$$L = K - P$$

$L \rightarrow$ lagrangian

K – Kinetic energy

P – Potential energy

Equation of motion

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k} \qquad k = 1, 2, 3, \ldots$$

q_k — generalised coordinate (~ degree of freedom)

τ_k – generalised force associated with q_k

Redo ball falling under gravity using Euler-Lagrange method

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$P = -mgy$$

$$\rightarrow L = K - P = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k$$

$q_k = x \quad \tau_k = F_x$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_x$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{\cancel{2}} m (\cancel{2}\dot{x})\right) - 0 = F_x$$

$$\Rightarrow m\ddot{x} = F_x = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}$$

$$\Rightarrow \boxed{\ddot{x} = -\frac{c}{m}\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}} \quad \text{— IV}$$

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$\Rightarrow q_k = y \qquad \tau_k = F_y$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = F_y$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{\cancel{2}} m \left(\cancel{2}\dot{y}\right)\right) - (-mg) = F_y$$

$$\Rightarrow m\ddot{y} + mg = F_y = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}$$

$$\Rightarrow \boxed{\ddot{y} = -g - \frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}} \qquad \text{V}$$

comparing IV & V with III & IV we see that Euler-lagrange gives the same answer as Newton's method.

Simulation and animation in MATLAB

$$\ddot{x} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\;\dot{x}$$

$$\ddot{y} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\;\dot{y} - g$$

Ordinary Differential equations

We also need to specify initial conditions

At time $t=0$ $\quad x_0, y_0, \dot{x}_0, \dot{y}_0$ are given

Implementation in MATLAB:

$[t, q] = \text{ode45}(@\text{equations}, \text{tspan}, q_0, \text{options}, \text{parameters})$

tolerances

time

state $x, y, \dot{x}, \dot{y}$ at those times

function that gives $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$

time $(0, t_{end})$

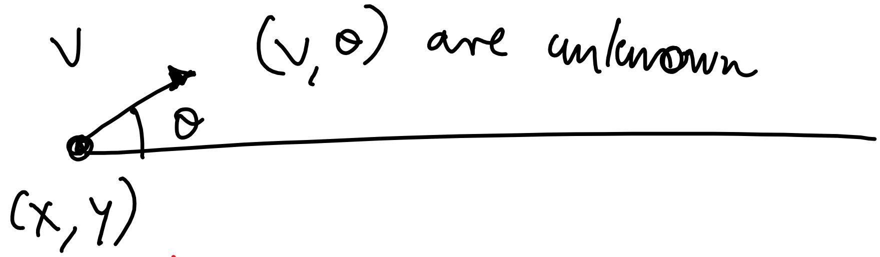
initial conditions $[x_0, \dot{x}_0, y_0, \dot{y}_0]$

additional parameters c, m, g.

Launch a projectile to hit a target

time t is given

Target (x_d, y_d) given



(v, θ) are unknown

(x, y) given

This problem is similar to the inverse kinematics problem of the manipulator or differential drive car (e.g. tracing an astroid curve)

$$q_0 = \text{fsolve}(\text{'constraints'}, \text{q\_guess}, \text{options}, \text{parameters})$$

solution v, θ

describe the target $\begin{bmatrix} x_t - x_d \\ y_t - y_d \end{bmatrix}$

initial guess for v, θ