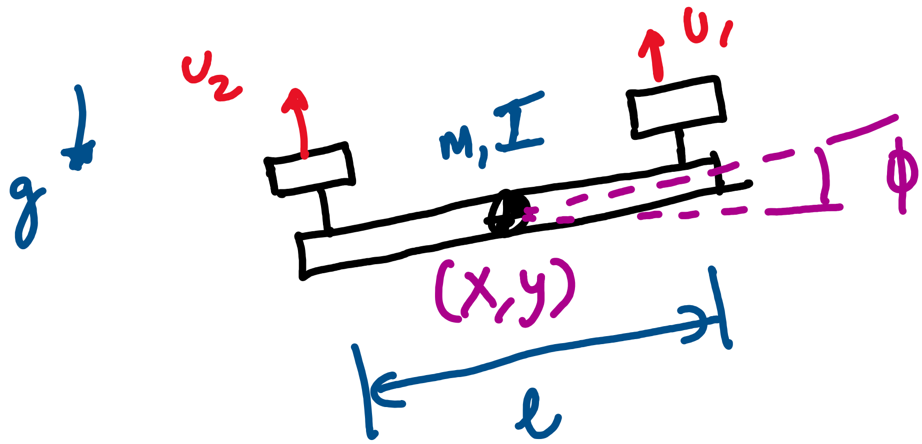
Bicopter

2D version of a quadcopter



m, I mass, inertia
g, l gravity, length
u_1, u_2 thrust forces 2 actuator
x, y, ϕ degrees of freedom 3 dofs

Equations of motion

① Get position/velocities of the center of mass.

x, y, ϕ positions
$\dot{x}, \dot{y}, \dot{\phi}$ velocities.

② Compute kinetic / Potential energy

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\phi}^2$$

$$V = mgy$$

$$\mathcal{L} = T - V$$

$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\phi}^2 - mgy$$

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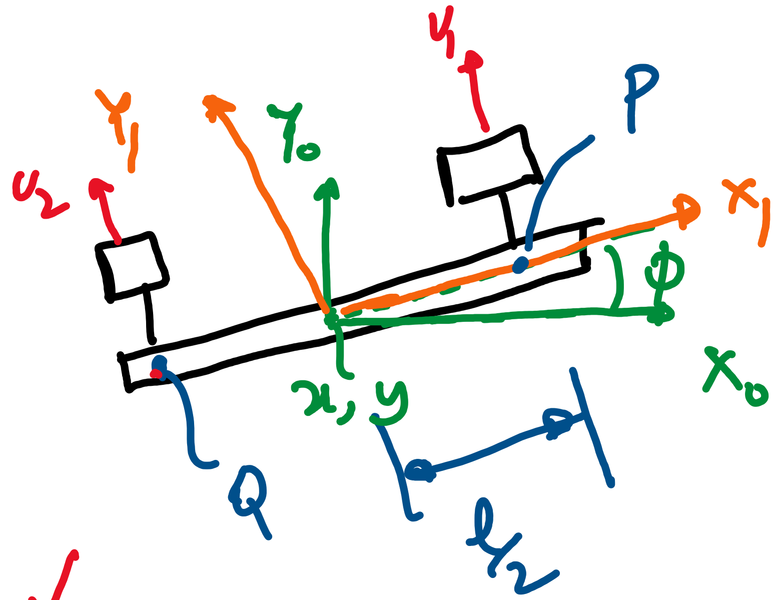
③ Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$q_j = x, y, \phi$$

$$Q_j = F_x, F_y, \tau_\phi$$

Lets compute Q_j



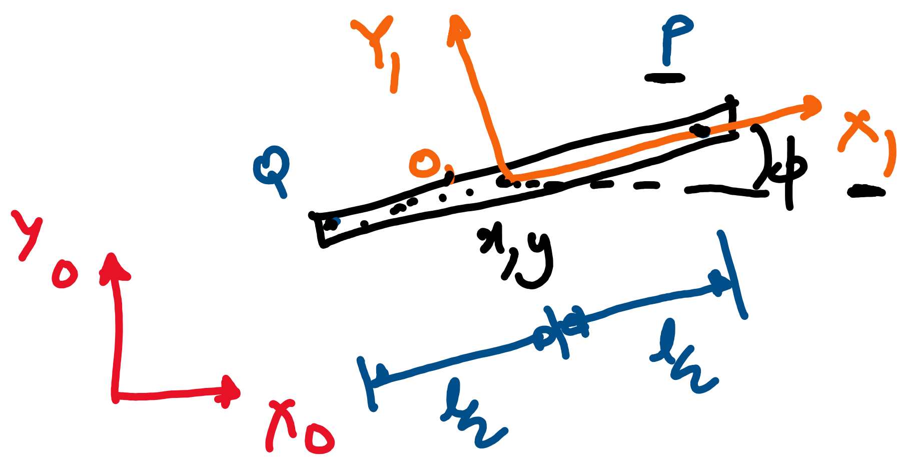
$$Q_j = J_P^T F_P^0 + J_Q^T F_Q^0$$

$$J_P = \frac{\partial r_P^0}{\partial q} \qquad J_Q = \frac{\partial r_Q^0}{\partial q}$$

$$r_P^0 = [x_P^0, y_P^0]$$

$$r_Q^0 = [x_Q^0, y_Q^0]$$

$$q = [x, y, \phi]$$



$$H_1^0 = \begin{bmatrix} R_1^0 & \underline{O}_1^0 \\ \underline{0} & \underline{1} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^0 = H_1^0 P^1 = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x + l/2 \cos\phi \\ y + l/2 \sin\phi \\ 1 \end{bmatrix} = \begin{bmatrix} x_P^0 \\ y_P^0 \\ 1 \end{bmatrix}$$

$$Q^0 = H_1^0 Q^1 = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x - l/2 \cos\phi \\ y - l/2 \sin\phi \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x_Q^0} \\ y_Q^0 \\ 1 \end{bmatrix}$$

$$J_P^0 = \begin{bmatrix} \frac{\partial x_P^0}{\partial x} & \frac{\partial x_P^0}{\partial y} & \frac{\partial x_P^0}{\partial \phi} \\ \frac{\partial y_P^0}{\partial x} & \frac{\partial y_P^0}{\partial y} & \frac{\partial y_P^0}{\partial \phi} \end{bmatrix}$$

$$x_P^0 = x + \frac{l}{2}\cos\phi \; ; \quad y_P^0 = y + \frac{l}{2}\sin\phi$$

$$J_P^0 = \begin{bmatrix} 1 & 0 & 0 + \frac{l}{2}(-\sin\phi) \\ 0 & 1 & 0 + \frac{l}{2}\cos\phi \end{bmatrix}_{2\times 3}$$

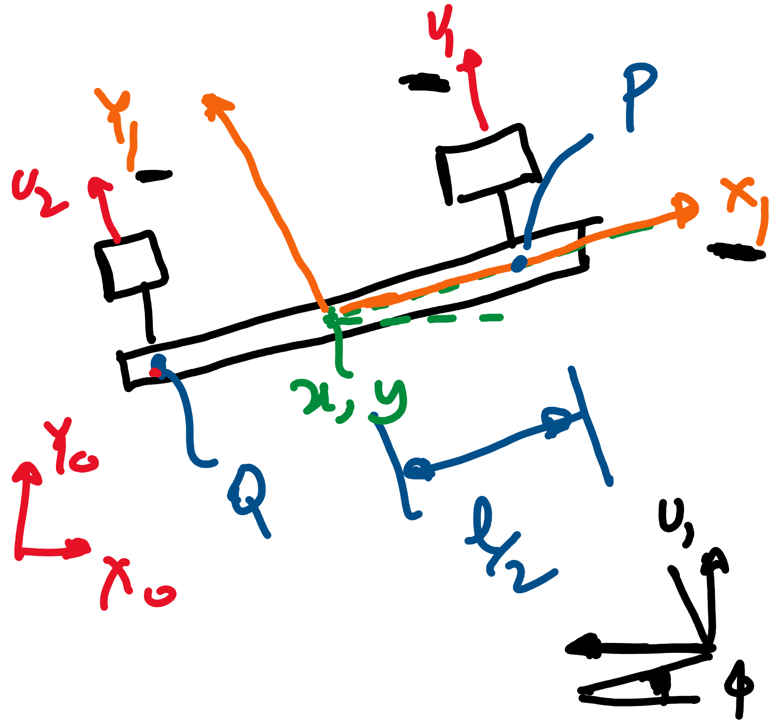
$$J_Q^0 = \begin{bmatrix} \frac{\partial x_Q^0}{\partial x} & \frac{\partial x_Q^0}{\partial y} & \frac{\partial x_Q^0}{\partial \phi} \\ \frac{\partial y_Q^0}{\partial x} & \frac{\partial y_Q^0}{\partial y} & \frac{\partial y_Q^0}{\partial \phi} \end{bmatrix}$$

$$x_Q^0 = x - \frac{l}{2}\cos\phi \; ; \quad y_Q^0 = y - \frac{l}{2}\sin\phi$$

$$J_Q^0 = \begin{bmatrix} 1 & 0 & \frac{l}{2}\sin\phi \\ 0 & 1 & -\frac{l}{2}\cos\phi \end{bmatrix}$$

$$F_P^1 = \begin{bmatrix} 0 \\ u_1 \end{bmatrix}$$

$$F_Q^1 = \begin{bmatrix} 0 \\ u_2 \end{bmatrix}$$



$$F_P^0 = R_1^0 \, F_P^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \end{bmatrix}$$

$$= \begin{bmatrix} -u_1 \sin\phi \\ u_1 \cos\phi \end{bmatrix}$$

$$F_Q^0 = R_1^0 \, F_Q^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} -u_2 \sin\phi \\ u_2 \cos\phi \end{bmatrix}$$

$$Q_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -l_2 \sin\phi & l_2 \cos\phi \end{bmatrix} \begin{bmatrix} U_1 \sin\phi \\ U_1 \cos\phi \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ l_2 \sin\phi & -l_2 \cos\phi \end{bmatrix} \begin{bmatrix} -U_2 \sin\phi \\ U_2 \cos\phi \end{bmatrix}$$

$$Q_j = \begin{bmatrix} -(U_1 + U_2) \sin\phi \\ (U_1 + U_2) \cos\phi \\ (U_1 - U_2)(0.5\, l) \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ \tau_\phi \end{bmatrix}$$

$$\mathcal{L} = 0.5\, m(\dot{x}^2 + \dot{y}^2) + 0.5\, I \dot{\phi}^2 - mgy$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$q_j = x$

$$\frac{d}{dt}\left(0.5\, m\,(2\dot{x})\right) - 0 = F_x = -(u_1+u_2)\sin\phi$$

$$\ddot{x} = -\frac{(u_1+u_2)}{m}\sin\phi \qquad (1)$$

$q_j = y$

$$\frac{d}{dt}\left(0.5\, m\,(2\dot{y})\right) - (-mg) = F_y = (u_1+u_2)\cos\phi$$

$$\ddot{y} = -g + \frac{(u_1+u_2)}{m}\cos\phi \qquad (2)$$

$q_j = \phi$

$$\frac{d}{dt}\left(0.5\, I\,(2\dot{\phi})\right) - 0 = \tau_\phi = (u_1-u_2)\,0.5\,l$$

$$\ddot{\phi} = (u_1-u_2)\frac{0.5\,l}{I} \qquad (3)$$

Bicopter Equations

$$\ddot{x} = -\frac{U_s}{m} \sin\phi$$

$$U_s = U_1 + U_2$$

$$\ddot{y} = \frac{U_s}{m} \cos\phi - g$$

$$U_d = U_1 - U_2$$

$$\ddot{\phi} = \frac{0.5\, l}{I} U_d$$

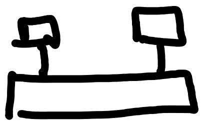
3 dofs (x, y, ϕ)

2 controls U_s, U_d or U_1, U_2

$$\ddot{x} = -\frac{U_s}{m} \sin\phi$$
$$\ddot{y} = \frac{U_s}{m} \cos\phi - g$$
$$\ddot{\phi} = \frac{0.5l}{I} U_d$$

Intuition

① Hover



$\phi = 0$

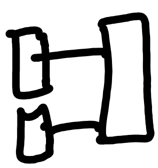
$\ddot{x} = 0$

$\ddot{y} = \frac{U_s}{m}(1) - g = 0$ $\quad U_s = mg = V_1 + V_2$

$\ddot{\phi} = \frac{0.5l}{I}(0)$ $\quad U_d = U_1 - U_2$

$\Rightarrow U_1 = U_2$

② $\phi = 90°$

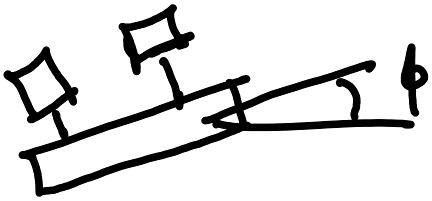


$\ddot{x} = -\frac{U_s}{m}(1)$

$\ddot{y} = 0 - g$ ← crash
No control.

(3) $0 \leq \phi \leq 90^\circ$

$$\ddot{x} = -\frac{U_s}{m}\sin\phi$$
$$\ddot{y} = \frac{U_s}{m}\cos\phi - g$$
$$\ddot{\phi} = \frac{0.5l}{I}U_d$$



$\phi > 0$

$\phi \approx 0.1$

$$\ddot{x} \approx -\frac{U_s}{m}\phi = -g\phi$$

Hover $U_s \approx mg$

$$\ddot{y} \approx \frac{U_s}{m} - g = \frac{mg}{m} - g \approx 0$$

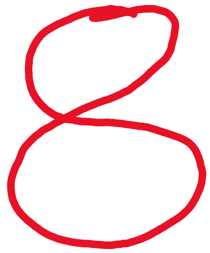
– stationary

$$\ddot{\phi} = \frac{0.5l}{I}U_d$$

$\ddot{x}$ = -ive

move this way

Feedback linearization for trajectory tracking



x_{ref}, y_{ref} ✓

$\dot{x}_{ref}, \dot{y}_{ref}$ ✓

$\ddot{x}_{ref}, \ddot{y}_{ref}$ ✓

$$\phi_{ref} = -\frac{1}{g}\left[\ddot{x}_{ref} + k_{px}(x_{ref} - x) + k_{dx}(\dot{x}_{ref} - \dot{x})\right]$$

$$u_s = mg + \delta u_s$$

$$\delta u_s = m\left(\ddot{y}_{ref} + k_{py}(y_{ref} - y) + k_{dy}(\dot{y}_{ref} - \dot{y})\right)$$

$$u_d = \delta u_d$$

$$\delta u_d = -k_{d\phi}\dot{\phi} - k_{p\phi}(\phi_{ref} - \phi)$$ ✓