Linearization

Pole placement / LQR only works for a linear system of the form

$$\dot{x} = Ax + Bu$$

But most systems are non-linear

$$\dot{x} = f(x, u)$$

where f is non-linear

We will linearize the system:

$$\dot{x} = f(x,u) \xrightarrow[\text{Linearization}]{} \dot{x} = Ax + Bu$$

Linearization using the Taylor series

x_0, u_0 – operating point

$\dot{x} = f(x, u)$ – system dynamics

$\boxed{\dot{x}_0 = f(x_0, u_0) \quad \text{True}}$ – ①

$x = x_0 + \delta x$

$u = u_0 + \delta u$

$\dot{\overline{(x_0 + \delta x)}} = f(x_0 + \delta x, u_0 + \delta u)$

$$\dot{x}_0 + \delta\dot{x} = f(x_0, u_0) + \frac{\partial f}{\partial x}\overbrace{(x - x_0)}^{\delta x} + \frac{\partial f}{\partial u}\overbrace{(u - u_0)}^{\delta u} +$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\overbrace{(x - x_0)}^{\delta x}{}^2 + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}\overbrace{(u - u_0)}^{\delta u}{}^2 + \text{high order terms in } \delta x, \delta u$$

$$\dot{x}_0 + \delta\dot{x} = f(x_0, u_0) + \frac{\partial f}{\delta x}\delta x + \frac{\partial f}{\partial u}\delta u + \cancel{HOT}$$

0

Since $\delta x, \delta u$ is small.

$$\cancel{\dot{x}_0} + \delta\dot{x} = \cancel{f(x_0, u_0)} + \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \delta u$$

$$\dot{x}_0 = f(x_0, u_0)$$

$$\underbrace{\delta\dot{x}}_{X} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}}_{A} \underbrace{\delta x}_{X} + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}}_{B} \underbrace{\delta u}_{u}$$

$$\delta x = X \; ; \; \delta u = u \; ;$$

$$\boxed{\left.\frac{\partial f}{\delta x}\right|_{x_0,u_0} = A \; ; \; \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} = B}$$

$$\dot{X} = AX + Bu$$

Linearization

$$u = -KX$$

$$\boxed{\delta u = -K\,\delta x}$$ — controller

$$\boxed{u = u_0 + \delta u}$$ — goes to the motors

EXAMPLE:

For the system
$$\dot{x} = V\cos\theta$$
$$\dot{y} = V\sin\theta$$
$$\dot{\theta} = \omega$$

Linearize it about an operating point $z_0 = [x_0, y_0, \theta_0]$; $u_0 = [V_0, \omega_0]$

$$\dot{\underline{z}} = f(z, u)$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} V\cos\underline{\theta} \\ V\sin\underline{\theta} \\ \omega \end{bmatrix}}_{f(z,u)}$$

non-linear due to $\cos$, $\sin$; $V\cos$, $V\sin$

$$A = \left.\frac{\partial f}{\partial z}\right|_{z_0, u_0} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{z_0, u_0}$$

$$A = \frac{\partial f}{\partial z} = \frac{\partial}{\partial(x,y,\theta)} \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} v\cos\theta & \frac{\partial}{\partial y} v\cos\theta & \frac{\partial}{\partial \theta} v\cos\theta \\ \frac{\partial}{\partial x} v\sin\theta & \frac{\partial}{\partial y} v\sin\theta & \frac{\partial}{\partial \theta} v\sin\theta \\ \frac{\partial}{\partial x} \omega & \frac{\partial}{\partial y} \omega & \frac{\partial}{\partial \theta} \omega \end{bmatrix}$$

$$A = \frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & -V_0 \sin\theta_0 \\ 0 & 0 & V_0 \cos\theta_0 \\ 0 & 0 & 0 \end{bmatrix}$$ ANSWER

$$B = \frac{\partial f}{\partial u} = \frac{\partial}{\partial (v,\omega)} \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial v} v\cos\theta & \frac{\partial}{\partial \omega} v\cos\theta \\ \frac{\partial}{\partial v} v\sin\theta & \frac{\partial}{\partial \omega} v\sin\theta \\ \frac{\partial}{\partial v} \omega & \frac{\partial}{\partial \omega} \omega \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix}$$

ANSWER

$$\delta \dot{z} = \begin{bmatrix} 0 & 0 & -v_0\cos\theta_0 \\ 0 & 0 & v_0\sin\theta_0 \\ 0 & 0 & 0 \end{bmatrix} \delta z + \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$