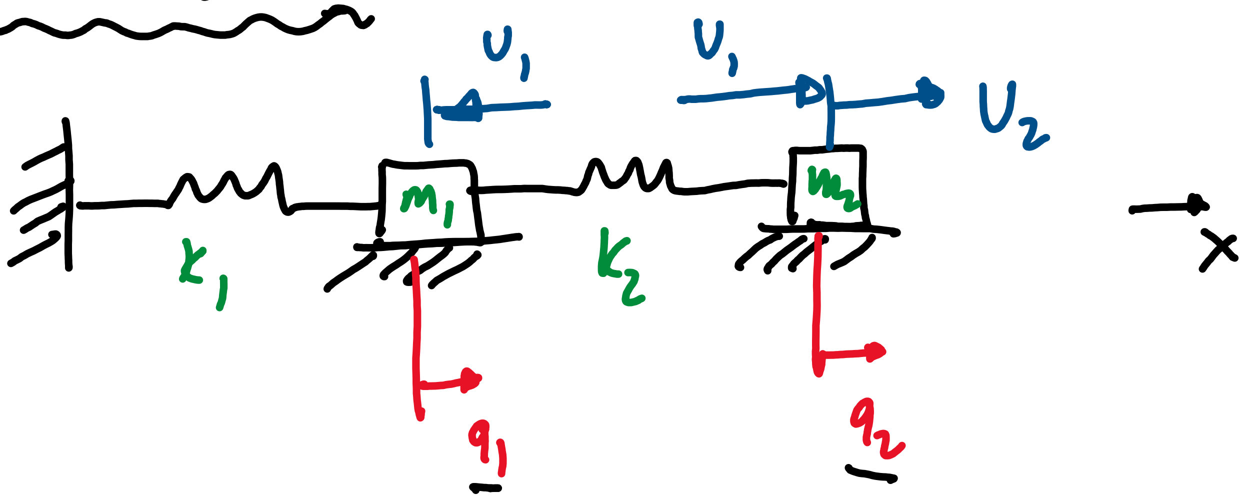
Linear control



① Derive the equations
② Use Linear control. (in time domain)

Most UG courses do this in frequency domain

① Derive the equations of motion

$$T = 0.5\, m_1 \dot{q}_1^2 + 0.5\, m_2 \dot{q}_2^2$$

$$V = 0.5\, k_1 q_1^2 + 0.5\, k_2 (q_1 - q_2)^2$$

$$\mathcal{L} = T - V$$

$$= 0.5\, m_1 \dot{q}_1^2 + 0.5\, m_2 \dot{q}_2^2 +$$
$$0.5\, k_1 q_1^2 + 0.5\, k_2 (q_1 - q_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = -U_1$$

$$m_1 \ddot{q}_1 + k_1 q_1 - k_2 (q_1 - q_2) = -U_1$$

$$\boxed{\ddot{q}_1 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) q_1 + \left(\frac{k_2}{m_1}\right) q_2 - \frac{U_1}{m_1}}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = U_1 + U_2$$

$$m_2 \ddot{q}_2 - k_2 (q_1 - q_2) = U_1 + U_2$$

$$\boxed{\ddot{q}_2 = \frac{k_2 q_1}{m_2} - \frac{k_2 q_2}{m_2} + \frac{U_1}{m_2} + \frac{U_2}{m_2}}$$

$$x_1 = q_1$$

$$x_2 = q_2$$

$$x_3 = \dot{q}_1$$

$$x_4 = \dot{q}_2$$

$$\dot{x}_1 = x_3 \quad \checkmark$$

$$\dot{x}_2 = x_4 \quad \checkmark$$

$$\dot{x}_3 = -\frac{(k_1+k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2 - \frac{u_1}{m_1}$$

$$\dot{x}_4 = \frac{k_2}{m_2} x_1 - \frac{k_2 x_2}{m_2} + \frac{u_1+u_2}{m_2}$$

$$\underset{4\times 1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}} = \underset{4\times 4}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} + \underset{4\times 2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{m_1} & 0 \\ \frac{1}{m_2} & \frac{1}{m_2} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

4×1 m_2 m_2 4×4 4×1 4×2

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\dot{X}}_{4\times 1} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & 0 & 0 \end{bmatrix}}_{A}_{4\times 4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X}_{4\times 1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{m_1} & 0 \\ \frac{1}{m_2} & \frac{1}{m_2} \end{bmatrix}}_{B}_{4\times 2} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}_{2\times 1}$$

$$\boxed{\dot{X} = AX + Bu}$$

Linear equation

State space representation.

Stability of the system (uncontrolled)

$$\dot{x} = Ax$$

To check stability

1) compute eigenvalues of A:

$$\det |A - \lambda I| = 0$$

and solve for λ (unknown, eigenvalues)

2) If the real part of the eigenvalues are negative, then the system is STABLE, else not.

Controllability

A linear system is controllable if and only if it can be transferred from any initial state $x = x_0$ to any terminal state $x = x_t$ in finite time.

Controllabity matrix (C_o)

$$C_o = [A^{n-1}B, A^{n-2}B, \dots, A^2B, AB, B]$$

here $\underline{n}$ is the dimension of X.

$$\underset{n\times 1}{\dot{X}} = \underset{n\times n}{A}\ \underset{n\times 1}{X} + \underset{n\times m}{B}\ \underset{m\times 1}{u}$$

\# state

\# controls

To check controllability

If rank $(C_o) = n$ system is controllable

rank $(C_o) < n$ system is uncontrollable

```
pip install control

Co = control.ctrb(A,B)

np.linalg.matrix_rank(Co)
```

Methods to control the system

① Pole placement

Assume $u = -Kx$

$$\dot{x} = Ax + Bu$$
$$= Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}x$$

$$\dot{x} = \bar{A}x$$

Compute K such that the eigenvalues of $\bar{A} = A - BK$ have negative real part

K = control.place (A, B, p)

✓ output

↑ poles/ location of eigenvalues/ user-chooses

② Linear Quadratic Regulator (LQR)

$$\boxed{\min J_{LQR} = \int_0^{\infty} \left[x^T Q x + u^T R u + 2x^T N u \right] dx}$$

cost

unconstrained opt.

User-chosen matrices

$$\boxed{\dot{x} = Ax + Bu}$$

x - state $(n \times 1)$

u - control $(m \times 1)$

Q - $n \times n$ $\left\{ \underset{1\times n}{x^T} \underset{n \times n}{Q} \underset{n \times 1}{x} \right.$ scalar $\left. \right\}$ 1×1

R - $m \times m$ $\left\{ \underset{1\times m}{u^T} \underset{m \times m}{R} \underset{m \times 1}{u} \right.$ scalar $\left. \right\}$ 1×1

N - $n \times m$ $\left\{ \underset{1\times n}{x^T} \underset{n \times m}{N} \underset{m \times 1}{u} \right.$ scalar $\left. \right\}$ 1×1

There is an analytical solution the unconstrained optimization problem

$$u = -Kx \quad \text{(feedback form)}$$

$$K = -R^{-1}(B^T P + N^T) \quad — \circledast$$

$$\underbrace{A^T P + PA - (PB + N) R^{-1} (B^T P + N^T)}_{\text{Ricatti Equation}} + Q = 0$$

① solve for P using Ricatti eq^n

② Compute K from ⊛

K, P, E = control.lqr (A, B, Q, R, N)

$u = -Kx$ — poles — eigen values of $(A - BK)$

How to choose Q, R, N

$$J = \int_0^\infty \left(x^T Q x + u^T R u + 2x^T N u \right) dt$$

N = 0

$x^T Q x$ big Q more penalty on state deviation

$u^T R u$ big R more penalty on control.

$$Q = \begin{bmatrix} \frac{q_1}{\|\max(x_1)\|^2} & 0 & \cdots & \\ 0 & \frac{q_2}{\|\max(x_2)\|^2} & & \\ 0 & 0 & & \frac{q_n}{\|\max x_n\|^2} \\ 0 & 0 & & \end{bmatrix}$$

normalize

tune only n-numbers

$$R = \begin{bmatrix} \frac{r_1}{\|\max(v_1)\|^2} & 0 & \cdots & \\ 0 & \frac{r_2}{\|\max v_2\|^2} & & \\ 0 & \vdots & & \\ \vdots & & & \frac{r_n}{\|\max(v_n)\|^2} \end{bmatrix}$$

If all v's are torque

$$R = \begin{bmatrix} r_1 & 0 & \cdot & 0 \\ 0 & r_2 & & \\ & & r_3 & \\ & & & \ddots \\ 0 & \cdots & & r_n \end{bmatrix}$$

tune r_n numbers.