Extend the idea to 2D

1D: $m\ddot{q} + c\dot{q} + kq = F$

2D:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$F = -k_p q - k_d \dot{q}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -\underbrace{\begin{bmatrix} k_{p11} & k_{p12} \\ k_{p21} & k_{p22} \end{bmatrix}}_{\text{4 parameters}}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \underbrace{\begin{bmatrix} k_{d11} & k_{d12} \\ k_{d21} & k_{d22} \end{bmatrix}}_{\text{4 parameters}}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

8 parameters

To have a critical damped system e_{q_1}, e_{q_2} there will be 2 equations.

Free parameters = 8-2= 6 (Too many)

Feedback Linearization/ Control Partitioning

Feedback Linearization / Control partitioning

$q \in \mathbb{R}^n$

→ Dynamics

$$\underset{n\times n}{M(q)}\,\underset{n\times 1}{\ddot{q}} + C(q,\dot{q})\,\underset{n\times 1}{\dot{q}} + \underset{n\times 1}{G(q)} = \underset{n\times 1}{\tau} \quad -(1)$$

→ Feedback Control:

$$\tau = M(-k_p q - k_d \dot{q}) + C(q,\dot{q})\dot{q} + G(q) \quad -(2)$$

$q, \dot{q}$ – measured by onboard sensors.

Substitute (2) in (1)

$$M(q)\ddot{q} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{G(q)} = \overbrace{M(-k_p q - k_d \dot{q}) + \cancel{C(q,\dot{q})\dot{q}} + \cancel{G(q)}}^{\tau}$$

$$M[\ddot{q} + k_p q - k_d \dot{q}] = 0 \leftarrow$$

Since $M \neq 0$

$$\ddot{q} + k_p q - k_d \dot{q} = 0$$

$$\ddot{q} + K_p q + K_d \dot{q} = 0$$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} K_{p_1} & 0 & 0 & \cdots & 0 \\ 0 & K_{p_2} & 0 & \cdots & 0 \\ 0 & 0 & K_{p_3} & \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & & K_{p_n} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix} +$$

$$+ \begin{bmatrix} K_{d_1} & 0 & 0 & \cdots & 0 \\ 0 & K_{d_2} & 0 & \cdots & 0 \\ 0 & 0 & K_{d_3} & \cdots & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & & K_{d_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_n \end{bmatrix} = 0$$

These are n-decoupled equations

$$\left.\begin{aligned} \ddot{q}_1 + K_{p_1} q + K_{d_1} \dot{q} &= 0 \\ \ddot{q}_2 + K_{p_2} q + K_{d_2} \dot{q} &= 0 \\ &\vdots \\ \ddot{q}_n + K_{p_n} q + K_{d_n} \dot{q} &= 0 \end{aligned}\right\} \quad \underbrace{\ddot{q}_i + K_{p_i} q + K_{d_i} \dot{q} = 0}_{i = 1, \ldots n}$$

$$\boxed{m\ddot{q} + (c + k_d)\dot{q} + (k + k_p)q = 0 \qquad k_d = -c + 2\sqrt{(k+k_p)m}} \quad \text{—— } 1$$

1 dof

$$\ddot{q}_i + k_{d_i}\dot{q} + k_{p_i} q = 0 \quad \text{—— } 1^*$$

$m=1 \quad c=0 \quad k=0$ in $1 \rightarrow 1^*$

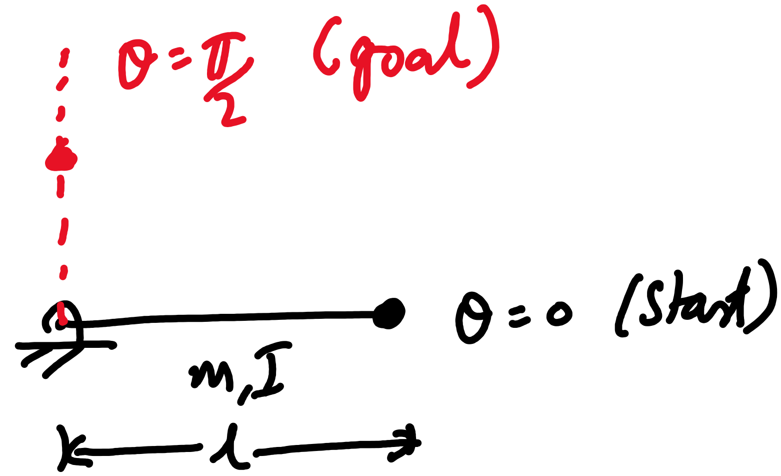
$$k_{d_i} = -c + 2\sqrt{(k+k_{p_i})m}$$

$$k_{d_i} = 0 + 2\sqrt{(0+k_{p_i})1}$$

$$\boxed{k_{d_i} = 2\sqrt{k_{p_i}}} \qquad i = 1, 2, \ldots n$$

{ See python file
2_control_partitioning_pd }

EXAMPLE: 1-link pendulum



$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$$(I + ml^2)\ddot{q} + 0 + mgl\sin(q) = \tau$$

Controllers

① $$\tau = -\underline{K_p}\,(q - \overset{\pi/2}{\overset{\parallel}{q_{des}}}) - \underline{K_d}\,\dot{q}$$

3a_pendulum_pd

② $$\tau = M\left(-K_p\,(q - \underset{\pi/2}{\underset{\parallel}{q_{des}}}) - \underline{K_d}\,\dot{q}\right) + mgl\sin(q)$$

3b_pendulum_control_partitioning.

$$K_d = 2\sqrt{K_p}$$