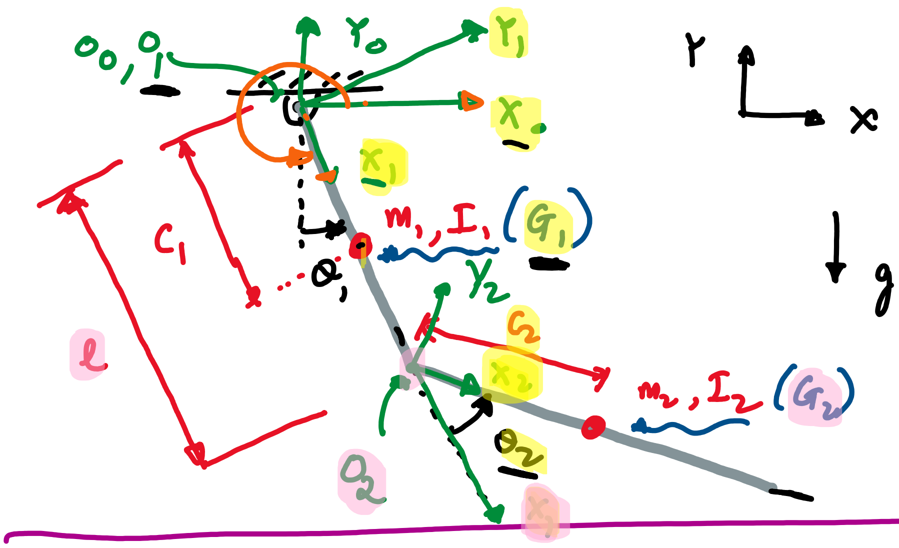
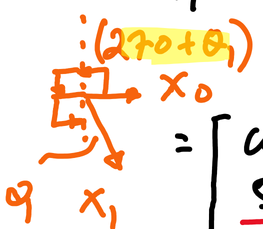
Double pendulum

Equations of motion, simulation & animation



① Position / Velocity of the center of mass.

$$G_1^0 = H_1^0 G_1^1 = \left[\begin{array}{c|c} R_1^0 & O_1^0 \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} g_1^1 \\ 1 \end{bmatrix}$$



$$= \left[\begin{array}{cc|c} \cos(270+\theta_1) & -\sin(270+\theta_1) & 0 \\ \sin(270+\theta_1) & \cos(270+\theta_1) & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} c_1 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \text{along } X_1 \\ \text{along } Y_1 \\ \end{array}$$

$$G_1^0 = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \\ 1 \end{bmatrix} \begin{array}{l} X_{G_1}^0 \\ Y_{G_1}^0 \\ \end{array}$$

$$G_2^0 = H_2^0 \, G_2^2 = H_1^0 \, H_2^1 \, G_2^2$$

$$H_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix} \checkmark \qquad H_2^1 = \left[\begin{array}{c|c} R_2^1 & o_2^1 \\ \hline 0 & 1 \end{array}\right]$$

$$\begin{array}{cl} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \text{along } x_1 \\ \text{along } y_1 \\ \end{array} \end{array}$$

$$G_2^2 = \begin{bmatrix} g_2^2 \\ 1 \end{bmatrix} = \begin{array}{cl} \begin{bmatrix} c_2 \\ 0 \\ 1 \end{bmatrix} & \begin{array}{l} \text{along } x_2 \\ \text{along } y_2 \\ \end{array} \end{array}$$

$$G_2^0 = \begin{array}{cl} \begin{bmatrix} l\sin\theta_1 + c_2 \sin(\theta_1+\theta_2) \\ -l\cos\theta_1 - c_2 \cos(\theta_1+\theta_2) \\ 1 \end{bmatrix} & \begin{array}{l} \underline{x_{G_2}^0} \\ \underline{y_{G_2}^0} \\ \end{array} \end{array}$$

Compute $V_{G_1}^0$, $V_{G_2}^0$.

Method 1 : Use Jacobian $V_{G_1}^0 = J_{G_1} \dot{q}$

$$V_{G_2}^0 = J_{G_2} \dot{q}$$

See Lec 08 b.

Method 2 $\quad V_{G_1}^{o} = \frac{d}{dt} g_1^{o}$; $V_{G_2}^{o} = \frac{d}{dt} g_2^{o}$

$$g_1^{o} = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix} \qquad \frac{d}{dt} g_1^{o} = \frac{d}{dt} \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix}$$

$$\dot{g}_1^{o} = \begin{bmatrix} c_1 \cos\theta_1 \, \dot{\theta}_1 \\ -c_1 (-\sin\theta_1) \, \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} c_1 \cos\theta_1 \, \dot{\theta}_1 \\ c_1 \sin\theta_1 \, \dot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} V_{G_1 x}^{o} \\ V_{G_1 y}^{o} \end{bmatrix}$$

$$\dot{g}_2^{o} = \frac{d}{dt} g_2^{o} = \frac{d}{dt} \begin{bmatrix} l \sin\theta_1 + c_2 \sin(\theta_1 + \theta_2) \\ -l \cos\theta_1 - c_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

A

$$\dot{g}_2 = \begin{bmatrix} [c_2 \cos(\theta_1 + \theta_2) + l \cos\theta_1] \dot{\theta}_1 + c_2 \cos(\theta_1 + \theta_2) \dot{\theta}_2 \\ [c_2 \sin(\theta_1 + \theta_2) + l \sin\theta_1] \dot{\theta}_1 + c_2 \sin(\theta_1 + \theta_2) \dot{\theta}_2 \end{bmatrix}$$

B

$$= \begin{bmatrix} V_{G_2 x}^{o} \\ V_{G_2 y}^{o} \end{bmatrix}$$

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② Compute $\mathcal{L} = T - V$

$$T = \frac{1}{2} m_1 (v_{G_1}^{o})^2 + \frac{1}{2} m_2 (v_{G_2}^{o})^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$\left(v_{G_{1x}}^2 + v_{G_{1y}}^2\right)$$

$$\left[\left((c_1 \cos\theta_1 \dot{\theta}_1)^2 + (c_1 \sin\theta_1 \dot{\theta}_1)^2\right]$$

$$\dot{v}_{G_{2x}}^2 + v_{G_{2y}}^2$$

$$(A)^2 + (B)^2$$

$$V = m_1 \underline{g}\, y_{G_1}^{o} + m_2 g\, y_{G_2}^{o}$$

$$= m_1 g(-c_1 \cos\theta_1) + m_2 g\left(-l \cos\theta_1 - c_2 \cos(\theta_1 + \theta_2)\right)$$

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③ $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \quad \}$$

$q_j = \underline{\theta_1}, \underline{\theta_2} \qquad Q_j = 0$

④ $\begin{bmatrix} EOM[0] \\ EOM[1] \end{bmatrix}$ - python

$\Downarrow$

$$\boxed{\begin{aligned} M_{11}\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 &= \cancel{C_1} + \cancel{G_1} \\ M_{21}\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 &= C_2 + G_2 \end{aligned}}$$

— EOM(0)

— EOM(1)

C_1, C_2 — coriolis acceleration $\approx \dot{\theta}_1^2$

G_1, G_2 — gravity term $\approx g$

$M_{11}, M_{12}, M_{21}, M_{22}$ — only functions of θ_1, θ_2

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} C_1 + G_1 \\ C_2 + G_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$