Symbolic calculations

Motivation

Use a computer to symbolically compute

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

Symbolic derivatives

Hand calculation

$f_0 = x^2 + 2x + 1$

$\frac{df_0}{dx} = 2x+2$

$\frac{df_0}{dx}\Big|(x=1) = 2(1)+2$

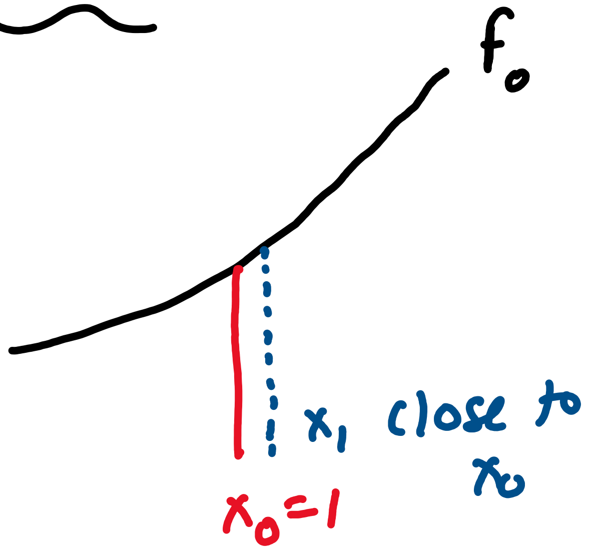
$= 4$

Python Symbolic

```
import sympy as sy

x = sy.sympy ("x", Real= "True")

f0 = x ** 2 + 2*x +1

df0dx = sy. diff (f0 ,x)

df0dx . subs (x, 1)
```

Python Numerical Derivative

$$\frac{df_0}{dx} = \frac{f_0(x_1) - f_0(x_0)}{x_1 - x_0}$$



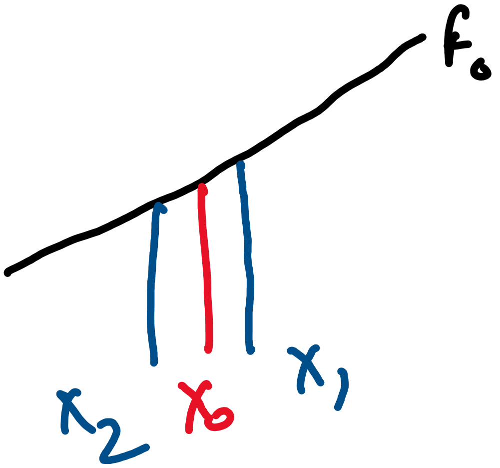
$x_1 = x_0 + 10^{-4}$

||

1e-4

Forward difference

$$\frac{df_0}{dx} = \frac{f_0(x_2) - f_0(x_1)}{(x_2 - x_1) = 2e^{-4}}$$



$x_1 = x_0 + 1e-4$

$x_2 = x_0 - 1e-4$

Central difference.

It is more accurate than forward difference for the same step size (e.g. 10^{-4})

Chain rule

If $f_1(x(t))$, compute $\frac{df_1}{dt}$

$$\begin{tikzcd} \underline{f_1} \arrow[r] \arrow[r, bend left] \arrow[rr, bend right] & \underline{x} \arrow[r] \arrow[r, bend left] & \underline{t} \end{tikzcd}$$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\frac{dx}{dt}$$ chain rule

EXAMPLE:

$$f_1 = \sin(x(t))$$

$$f_1 = \sin(x)$$

$$\frac{df_1}{dt} = \left(\frac{df_1}{dx}\right)\frac{dx}{dt}$$

$$= \frac{d(\sin(x)}{dx}\frac{dx}{dt}$$

$$\frac{df_1}{dt} = \cos x(t)\,\dot{x}$$

If $f_2(\underset{\substack{\uparrow \\ \text{position}}}{x}(t), \underset{\substack{\uparrow \\ \text{velocity}}}{\dot{x}}(t))$ then compute $\frac{df_2}{dt}$

$$\begin{tikzcd}
f_2 \arrow[r] \arrow[r, bend right] \arrow[rr, bend right] & x \arrow[rr, bend right] & \dot{x} \arrow[r] \arrow[r, bend right] & t
\end{tikzcd}$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

EXAMPLE:

$$f_2 = \underline{\underline{x(t)\,\dot{x}(t)}}$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$$= \frac{d}{dx}(x\dot{x})\underline{\frac{dx}{dt}} + \frac{d}{d\dot{x}}(x\dot{x})\frac{d\dot{x}}{dt}$$

$$= \dot{x}\;\dot{x} + x\;\ddot{x}$$

$$\boxed{\frac{df_2}{dt} = \dot{x}^2 + x\ddot{x}}$$

Back to Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = Q_j$$

Projectile : $\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$

$q_j = x, y$

$Q_j = F_{dx}, F_{dy}$

$$\mathcal{L}\,(x, y, \dot{x}, \dot{y})$$

$$\mathcal{L} \longrightarrow \boxed{\underline{x, y, \dot{x},}\ \dot{y}} \longrightarrow \underline{t}$$

$q_j = x$: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_{dx}$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \text{diff}(\mathcal{L}, \dot{x}) = \mathcal{L}_{\dot{x}}$ no chain rule

$\frac{\partial \mathcal{L}}{\partial x} = \text{diff}(\mathcal{L}, x)$ No chain rule

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}_{\dot{x}}}{\partial x}\underline{\frac{dx}{dt}} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial y}\underline{\frac{dy}{dt}} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial \dot{x}}\underline{\frac{d\dot{x}}{dt}} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial \dot{y}}\underline{\frac{d\dot{y}}{dt}}$$

chain rule