2D Dynamics

Newton's law (2nd law)

$F = ma$ $\qquad$ a = linear acceleration

$T = I\alpha$ $\qquad$ α = angular "

m = mass

I = inertia

F, T = Force, Torque

① Free Body Diagram ✗

② Use Newtons laws ✗

$F = ma$ / $T = I\alpha$

③ Given F, T, m, I ✓

solve for a, α — Equations of motion (EOM)

UGrad dynamics

Euler-Lagrange Method

- Compute EOM without using Free Body Diagrams

Algorithm for Euler-Lagrange Method

① Write down the equations for the position and velocity of the center of mass of objects/links with respect to world frame

Homogenous transformation

$$g^0 = H_1^0 g^1$$

Use Jacobian

$$V_G = J_G \dot{q}$$

② $\mathcal{L} = T - V$

$\mathcal{L}$ - lagrangian

$$T = \frac{1}{2} \sum_{i=1}^{n} (m_i v_i^2 + I_i \omega_i^2)$$ kinetic energy

v_i = linear speed

ω_i = angular speed

$$V = \sum_{i=1}^{n} m_i g_i y^0_{G_i} + \frac{1}{2}\sum_{i=1}^{q} k_{p_i}(r_{p_i} - r_{p_0})^2$$

g_i - gravity

k_{p_i} - spring constant

r_{p_i}, r_{p_0} - spring length, spring length in rest configuration

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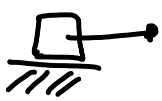
③ Equations of motion

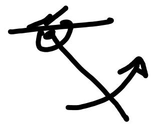
$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j}$$

Same as
$F = ma$
$T = I\alpha$

q_j = degrees of freedom



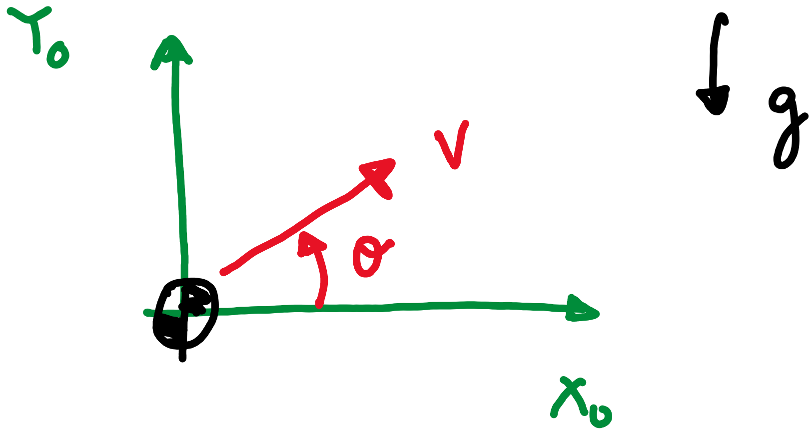
$q_j = x$



$q_j = \theta$

Q_j - External force including damping

EXAMPLE: Projectile motion under a quadratic drag force



Derive the equations of motion

Quadratic drag (F_d) : $\vec{F}_d = -c\,v^2\,\hat{v}$

v^2 - magnitude of the drag force

$\hat{v}$ - unit vector along velocity

$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$ $\quad$ $\hat{i}, \hat{j}$ unit vectors along x-, y-

$v^2 = |\vec{v}|^2 = \dot{x}^2 + \dot{y}^2$ $\quad$ c = constant

$$\hat{v} = \frac{\vec{v}}{|v|} = \frac{\dot{x}\,\hat{i} + \dot{y}\,\hat{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

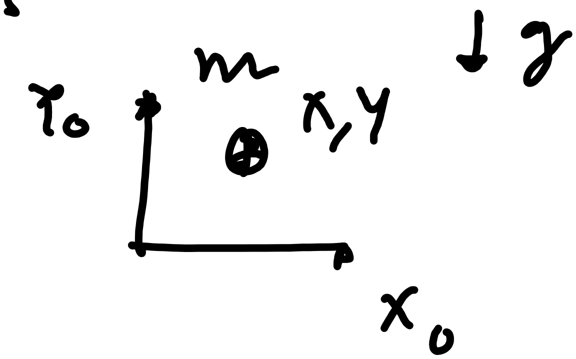
$$|v| \qquad \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$F_d = -c\underline{v}^2 \, |\hat{v}|$$

$$= -c\underline{(\dot{x}^2 + \dot{y}^2)} \left[\frac{\dot{x}\,\hat{i} + \dot{y}\,\hat{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right]$$

$$F_d = \underbrace{-c\sqrt{\dot{x}^2+\dot{y}^2}\;\dot{x}}_{F_{dx}}\,\hat{i} \; \underbrace{-\,c\sqrt{\dot{x}^2+\dot{y}^2}\;\dot{y}}_{F_{dy}}\,\hat{j}$$

Euler-Lagrange Equations



① positions: x, y

velocities: $\dot{x}, \dot{y}$

② $\mathcal{L} = T - V$

$$T = \frac{1}{2} m \underline{\underline{v^2}} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

③ $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$q_j = \underline{x}, y \;;\; Q_j = \underline{F_{dx}}, \underline{F_{dy}}$$

(i) $q_j = x$:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \underline{\dot{x}}}\left[\frac{m}{2}(\underline{\dot{x}}^2 + \underline{\dot{y}}^2) - mgy\right]\right) - \frac{\partial}{\partial x}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy\right] = F_{dx}$$

$$\frac{d}{dt}\left(\frac{m}{\cancel{2}}[\cancel{2}\dot{x} + 0] - 0\right) - 0 = F_{dx}$$

$$\frac{d}{dt}(m\underline{\dot{x}}) = F_{dx}$$

$$m\ddot{x} = F_{dx} = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}$$

$$\boxed{\ddot{x} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}}$$

(ii) $q_j = y$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy\right]\right) - \frac{\partial}{\partial y}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy\right] = F_{dy}$$

$$\frac{d}{dt}\left(\frac{m}{\cancel{2}}\left[0+\cancel{2}\dot{y}\right] - 0\right) - \left[0+0-mg\right] = F_{dy}$$

$$m\ddot{y} + mg = F_{dy} = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}$$

$$\boxed{\ddot{y} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y} - g}$$

④ Simulate the system : Integrate the equations of motion

h: step size

h fixed { (a) Euler's method

(b) Runge-Kutta's method (RK4)

h variable (c) Adaptive Runge-Kutta method

ode 45

z = odeint (odefn , init_cond , t , arguments)

python integration (C)

[x_0, v_{x_0}, y_0, v_{y_0}]

odefn : x, v_x, y, v_y

$v_x = \dot{x}$, $v_y = \dot{y}$

$$\boxed{\begin{aligned}
\dot{x} &= v_x \\
\dot{v}_x &= -\frac{c}{m}\sqrt{v_x^2+v_y^2}\; v_x \\
\dot{y} &= v_y \\
\dot{v}_y &= -\frac{c}{m}\sqrt{v_x^2+v_y^2}\; v_y - g
\end{aligned}}$$