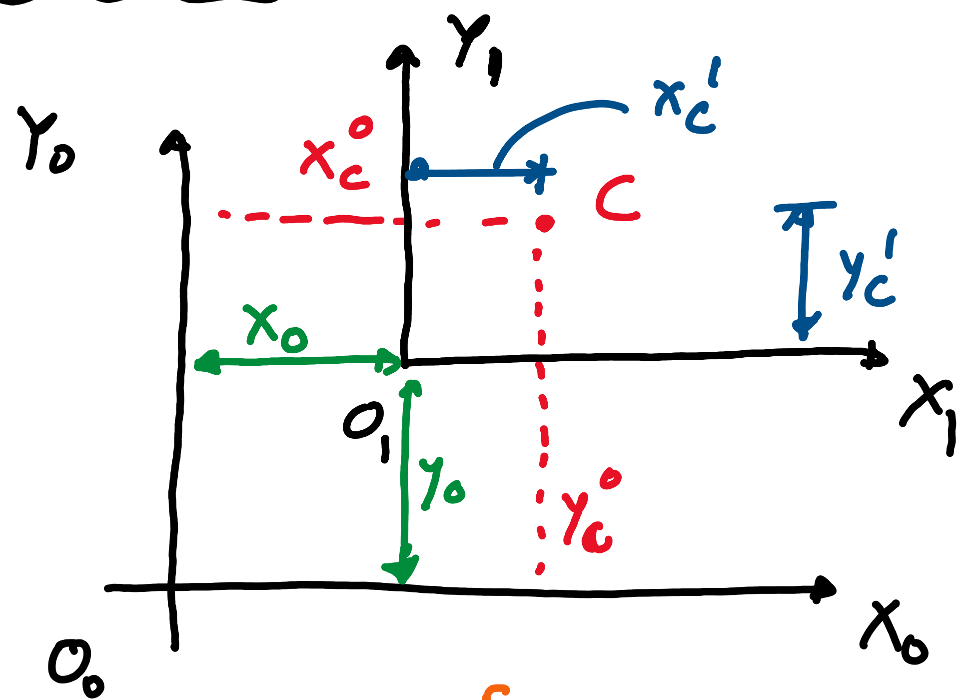
Coordinate Frames: Rotation & Translation

1.1 Translation

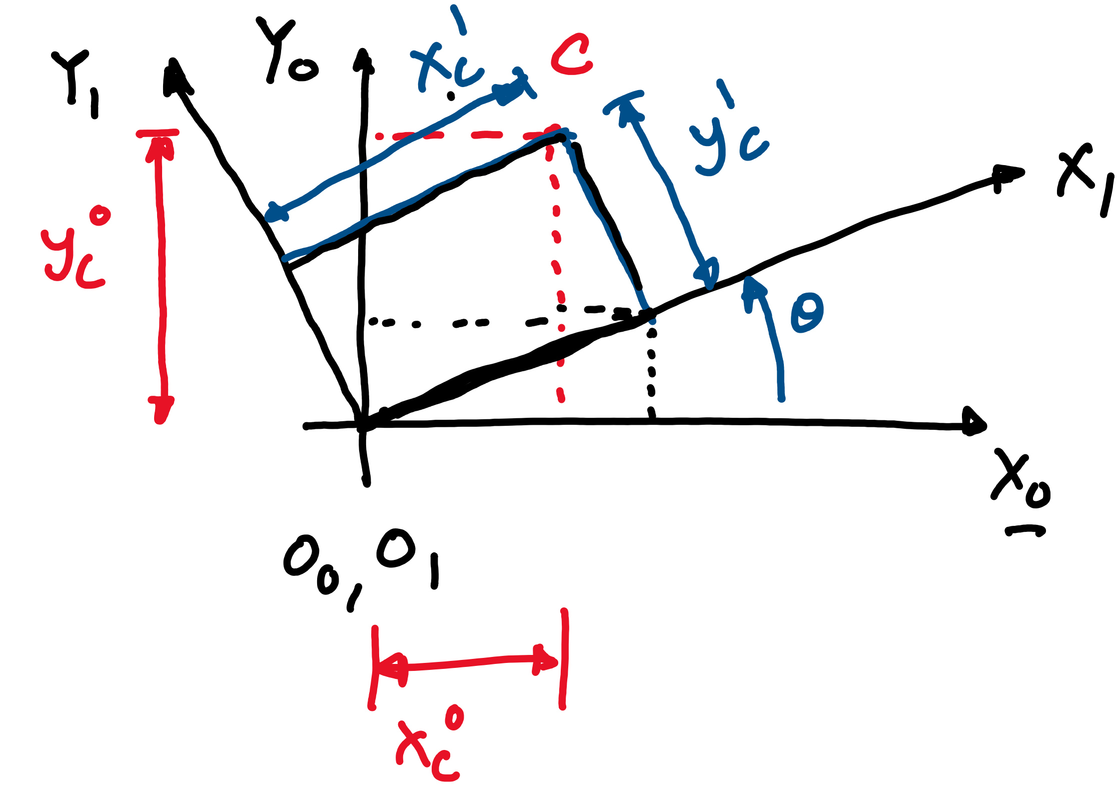


frame

$$C^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} \qquad C^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$

$$O_1^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad O_0^1 = \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix}$$

1.2 Rotation



$$x_c^0 = \cos\theta \, x_c^1 - \sin\theta \, y_c^1$$

$$y_c^0 = \sin\theta \, x_c^1 + \cos\theta \, y_c^1$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$

$$\boxed{c^0 = \underbrace{R_1^0}_{\text{rotation of frame 1 wrt. frame 0}} c_1}$$

$$\begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix}$$

$$\boxed{\underline{c}^1 = \underline{\underline{R}}_0^1 \; \underline{c}^0}$$

$$\left.\begin{aligned} R_1^0 &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ R_0^1 &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{aligned}\right\} \quad \boxed{R_1^0 = (R_0^1)^T}$$

Any rotation matrix has the property

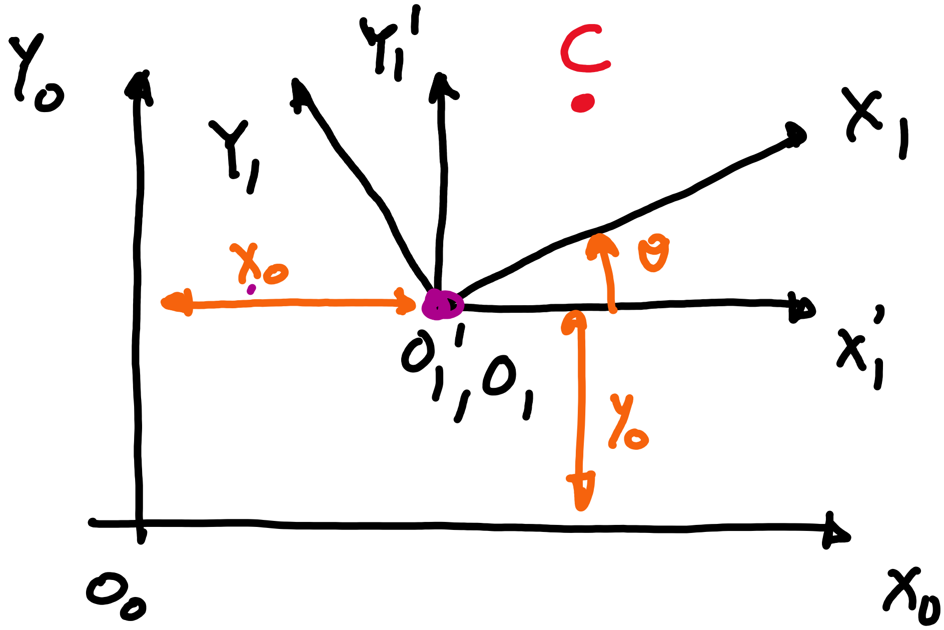
$$\boxed{R_1^0 (R_1^0)^T = I}$$

Pre-multiply with $(R_1^0)^{-1}$

$$\underbrace{(R_1^0)^{-1} (R_1^0)}_{I} (R_1^0)^T = (R_1^0)^{-1}$$

$$\boxed{(R_1^0)^{-1} = (R_1^0)^T}$$

1.3 Combined Rotation and Translation



$$O_0 X_0 Y_0 \longrightarrow O_1' X_1' Y_1' \longrightarrow O_1 X_1 Y_1$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \underbrace{\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}}_{1.1\ \text{Translation}} + \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c' \\ y_c' \end{bmatrix}}_{1.2\ \text{Rotation}}$$

1.3 Combined movement

$$\boxed{C^0 = O_1^0 + R_1^0 C^1}$$

1.4 Multiple Rotations / Translations

$O_0 X_0 Y_0 \rightarrow O_1 X_1 Y_1$

$$\underline{c}^0 = \underline{O_1^0 + R_1^0 \underline{c}^1} \quad — (1)$$

$O_1 X_1 Y_1 \rightarrow O_2 X_2 Y_2$

$$\underline{c}^1 = O_2^1 + R_2^1 \underline{c}^2 \quad — (2)$$

Combine (1) & (2)

$$c^0 = O_1^0 + R_1^0 \left[O_2^1 + R_2^1 c^2 \right]$$

$$c^0 = \underbrace{O_1^0 + R_1^0 O_2^1}_{\text{Translation}} + \underbrace{R_1^0 R_2^1}_{\text{Rotation}} c^2$$

Do this for n-frames

$O_0 X_0 Y_0 \rightarrow O_1 X_1 Y_1 \rightarrow \dots \rightarrow O_n X_n Y_n$

$$c^0 = \underbrace{\left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \dots + R_1^0 R_2^1 R_3^2 \dots O_n^{n-1}\right)}_{\text{Transl}} + \underbrace{R_1^0 R_2^1 R_3^2 \dots R_n^{n-1}}_{\text{Rotation}} c^n$$

Simple

Transl 3 $\underbrace{n_1 \quad n_2 \quad n_3 \quad \cdots \quad n_n}_{\text{Rotation}} \quad c$ =

1.5 Homogeneous Transformation (H)

Better way of tracking frames

$$H_i^{i-1} = \begin{bmatrix} \underbrace{[R_i^{i-1}]}_{2\times2} & \underbrace{[O_i^{i-1}]}_{2\times1} \\ \underbrace{[0]}_{1\times2} & \underbrace{1}_{1\times1} \end{bmatrix}_{3\times3}$$

e.g. $$H_1^0 = \begin{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} & \begin{bmatrix} x^0 \\ y^0 \end{bmatrix} \\ 0 \quad 0 & 1 \end{bmatrix}_{3\times3}$$

$$c^i = \begin{bmatrix} x_c^i \\ y_c^i \end{bmatrix} \qquad C^i = \begin{bmatrix} \left.\begin{matrix} x_{c_i}^i \\ y_c^i \end{matrix}\right\} c^i \\ 1 \end{bmatrix}_{3\times1}$$

$$C^{i-1} = H_i^{i-1} C^i$$

e.g. $i = 1 \Rightarrow C^0 = H_1^0 C^1$

$$\begin{bmatrix} c^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & o_1^0 \\ \underline{0} & 1 \end{bmatrix} \begin{bmatrix} c^1 \\ \underline{1} \end{bmatrix}$$

$$\begin{bmatrix} c^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 c^1 + o_1^0 \\ 0 + 1 \end{bmatrix}$$

$$\begin{bmatrix} c \\ 1 \end{bmatrix} = \begin{bmatrix} k_1 c + b, \\ 0 + 1 \end{bmatrix}$$

$$o_0 x_0 y_0 \longrightarrow o_1 x_1 y_1 \longrightarrow \cdots \quad o_n x_n y_n$$

$$\left.\begin{aligned} C^0 &= H_1^0 C^1 \\ C^1 &= H_2^1 C^2 \\ C_2 &= H_3^2 C^3 \\ &\vdots \\ C^{n-1} &= H_n^{n-1} C^n \end{aligned}\right\} \quad \boxed{C^0 = H_1^0 H_2^1 H_3^2 \cdots H_n^{n-1} C^n}$$

Same as 3