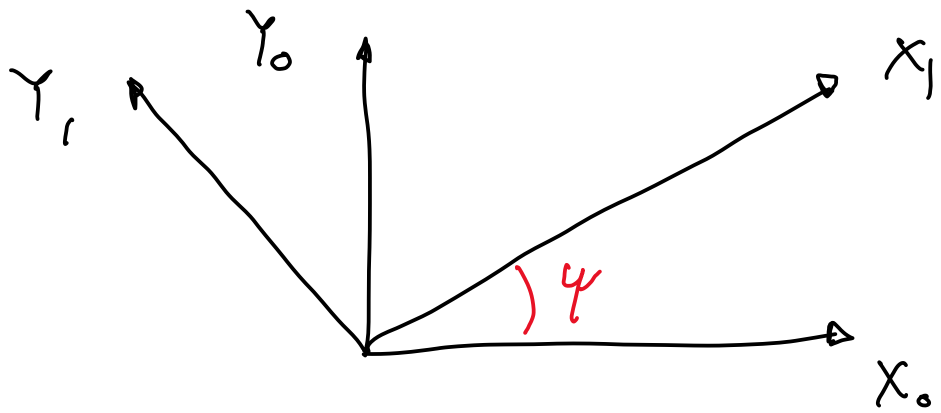
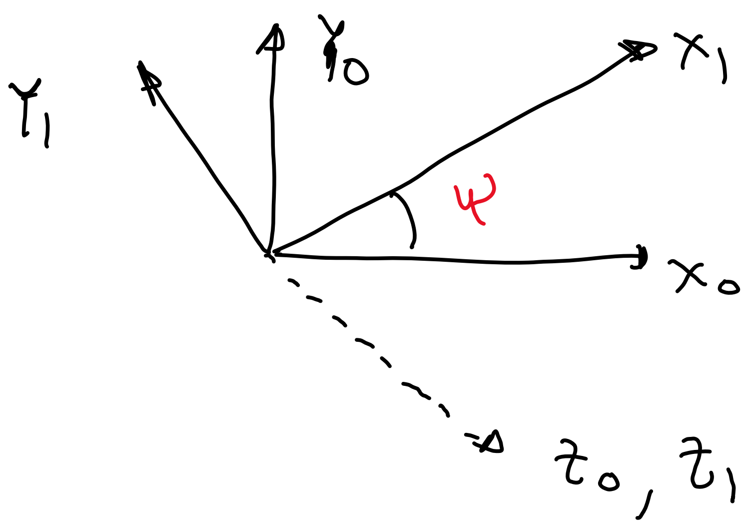
3D Rotations



2D $\quad R_1^0 = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}_{2\times 2}$



3D $\quad R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$z_0 = z_1$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

not an error

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

In general a rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$ 9 numbers

Certain properties restrict the range / values that r's in the Rotation matrix can take on

$$R^T R = I = R R^T \quad (I = 3\times3 \text{ identity matrix})$$

Scaled to 1 is encomposed in diagonal element

orthogonality is " " non-diagonal elements

$$\left.\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1 \end{aligned}\right\}$$

rotation matrix has unit magnitude

3 condition

$$\left.\begin{aligned} &\sum_{i=1,2,3} r_{i1} r_{i2} = 0 = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0 \\ &\sum_{i=1,2,3} r_{i2} r_{i3} = 0 \\ &\sum_{i=1,2,3} r_{i3} r_{i1} = 0 \end{aligned}\right\}$$

orthogonality of rotation matrix

3 conditions

9 parameters − 6 conditions = 3 free parameters.

or

3 unique numbers

EULER angles.

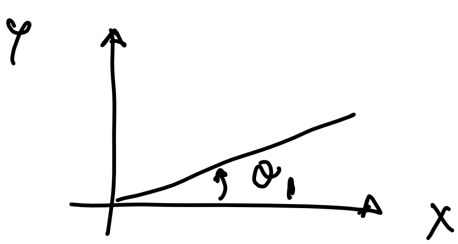
Euler Angles

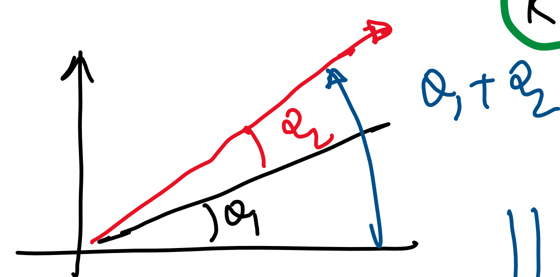
Tait-Bryan aerospace

1-2-3 Bryant angle

$\boxed{x-y-z}$	$y-x-z$	$\boxed{z-x-y}$
$x-z-y$	$y-z-x$	$\boxed{z-y-x}$ 3-2-1 we will use these in the course
$x-y-x$	$y-x-y$	$z-y-z$
$x-z-x$	$y-z-y$	$z-x-z$
4 +	4 +	4 = 12 unique ways of describing rotations.

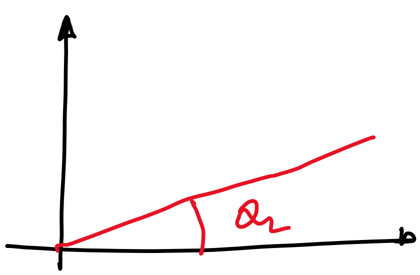
2D rotations commute

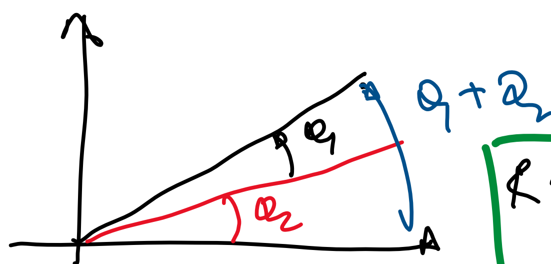




$$\boxed{R = R_{\theta_2} R_{\theta_1} = R_{\theta_1+\theta_2}}$$

|| same thing



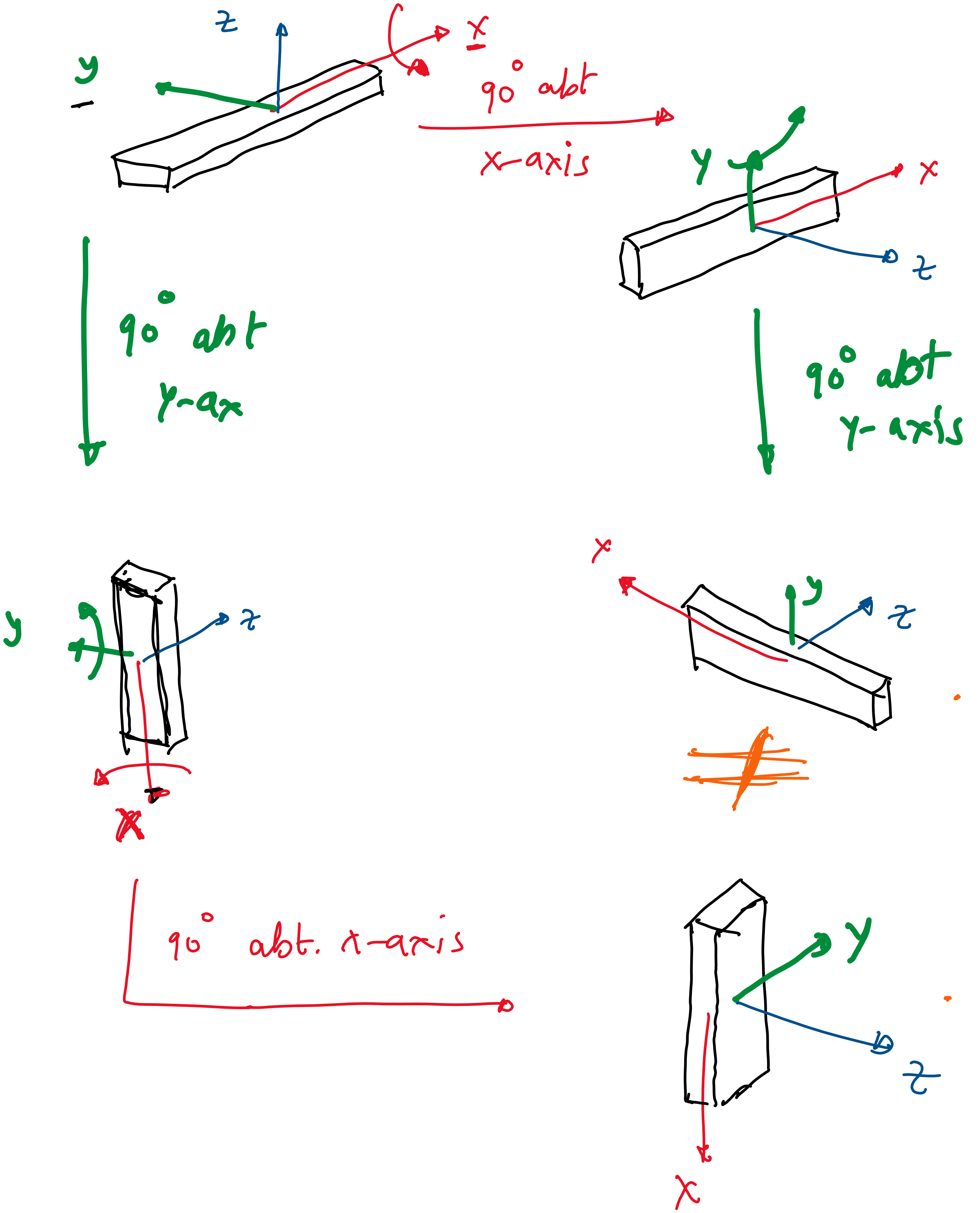


$$\boxed{R = R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}}$$

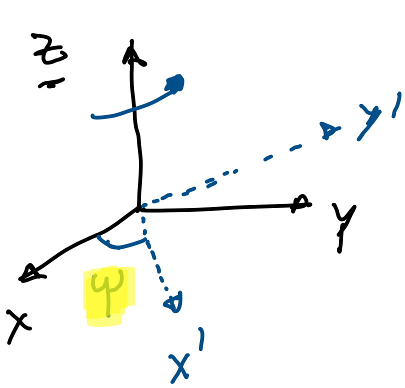
order of rotation is not important in 2D
$\Rightarrow$ rotations commute.

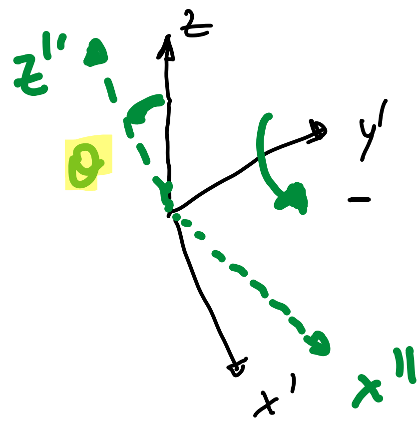
$\Rightarrow$ rotations commute.

Rotations in 3D do NOT commute



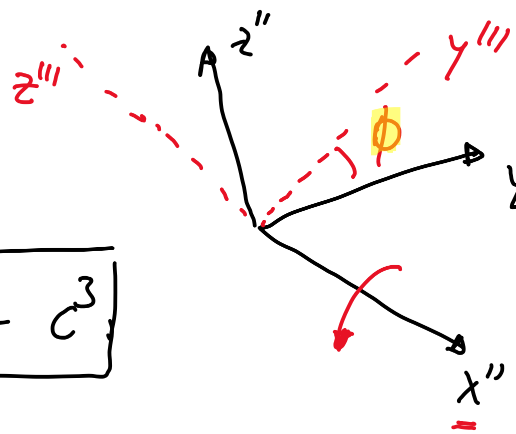
3-2-1 Euler Angles (z - y - x)





$\boxed{c^0 \longrightarrow c^1}$

$\boxed{c^1 \longrightarrow c^2}$



$\boxed{c^2 - c^3}$

$$c^0 = R_z(\psi)\, c^1$$

$$c^1 = R_y(\theta)\, c^2$$

$$c^2 = R_x(\phi)\, c^3$$

$$c^0 = \underline{R_z(\psi)\, R_y(\theta)\, R_x(\phi)}\, c^3$$

$$\underset{\text{fixed frame}}{c^0} = R \underset{\text{body frame}}{c^3} \Rightarrow \boxed{r^0 = R\, r^{body}}$$

r^0 - pos^n in fixed frame

r^{body} - pos^n in body frame

R - rotation matrix

corrected

$$R = R_z(\psi)\, R_y(\theta)\, R_x(\phi)$$

$$= \begin{bmatrix} \cos\psi \cos\theta & \cos\psi \sin\phi \sin\theta - \cos\phi \sin\psi & \sin\phi \sin\psi + \cos\psi \cos\phi \sin\theta \\ \cos\theta \sin\psi & \cos\psi \cos\phi + \sin\phi \sin\psi \sin\theta & \cos\phi \sin\psi \sin\theta - \cos\psi \sin\phi \\ -\sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta \end{bmatrix}$$

3X3