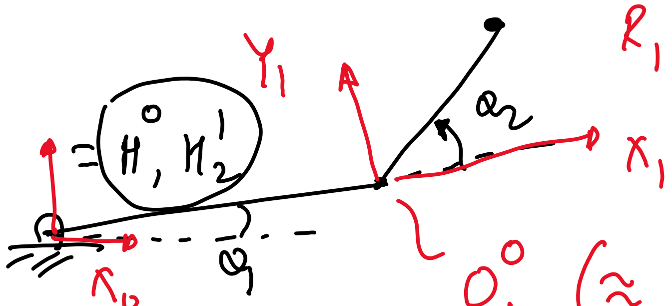
3D Manipulator Forward Kinematics

2D



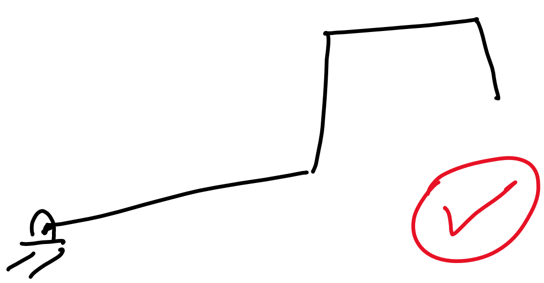
$R_1^0 \; (\approx R_i^{i-1})$

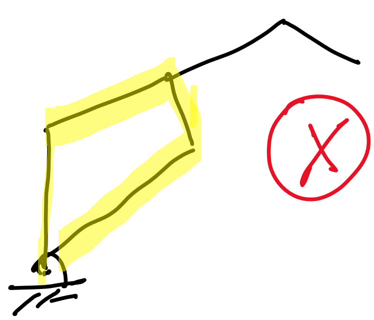
$O_1^0 \; (\approx O_i^{i-1})$

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1}{}_{\,2\times2} & O_i^{i-1}{}_{\,2\times1} \\ [0 \;\; 0]_{\,1\times2} & 1_{\,1\times1} \end{bmatrix}_{3\times3}$$

3D

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1}{}_{\,3\times3} & O_i^{i-1}{}_{\,3\times1} \\ [0 \;\; 0 \;\; 0]_{\,1\times3} & 1_{\,1\times1} \end{bmatrix}_{4\times4}$$

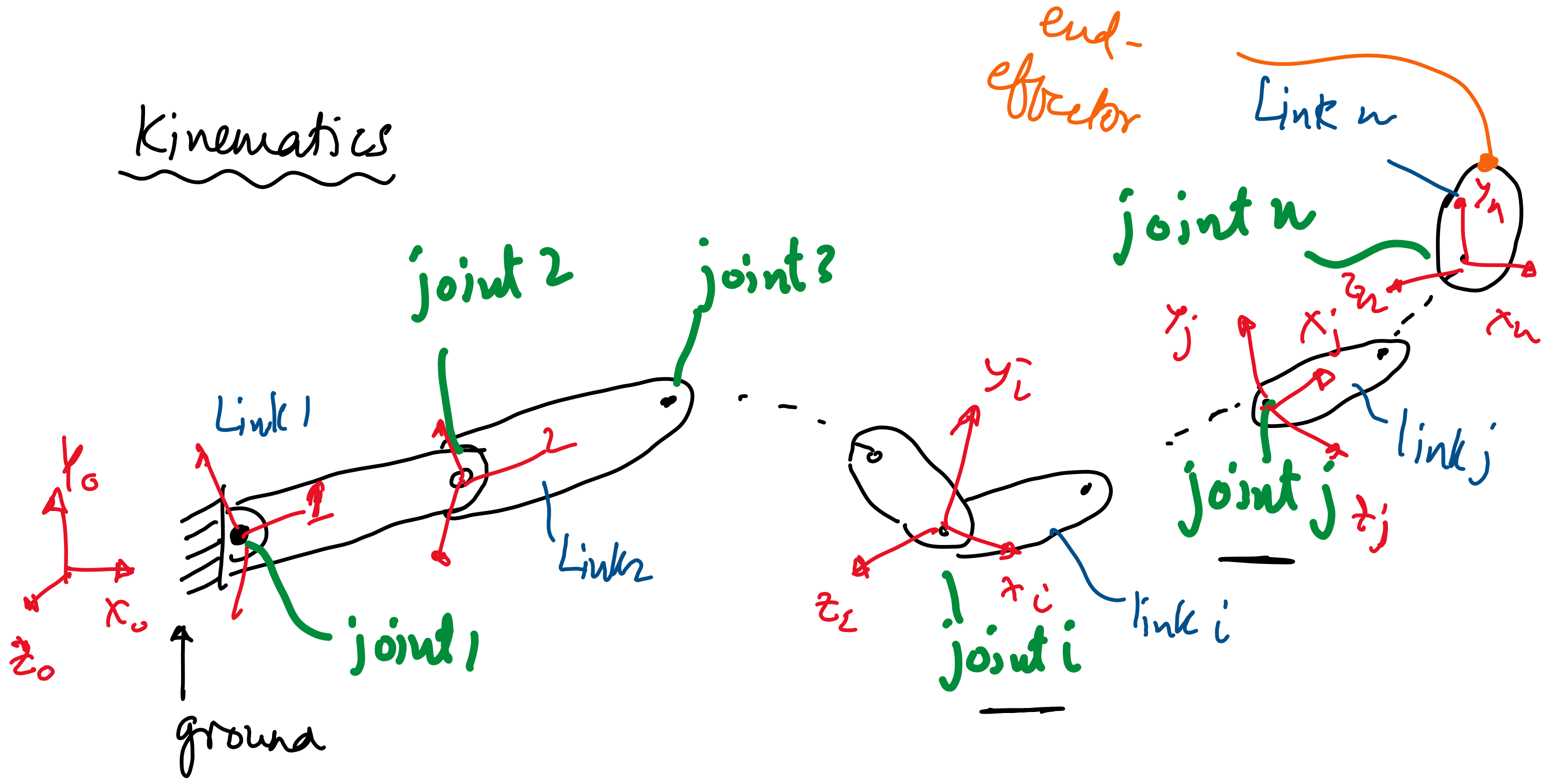




open loop

closed loop

Kinematics



$$H^i_j = H^i_{i+1} H^{i+1}_{i+2} \cdots H^{j-1}_j \qquad i < j$$

$$= I \qquad i = j$$

$$= \left(H^j_i\right)^{-1} \qquad i > j$$

$H \rightarrow$ 4x4 homogenous matrix.

$$R_x(\phi)$$

$$H_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

no translation

4X4

$$H_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Revolute joint motion

$$H_x(a_x) = \begin{bmatrix} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{=I} & \overbrace{\begin{pmatrix} a_x \\ 0 \\ 0 \end{pmatrix}}^{\text{- translation}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

$$H_y(a_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_z(a_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prismatic joint motion

Denavit-Hartenberg convention (DH)

Method to represent the kinematics of a manipulator.

$$H_i^{i-1} = H_z(\theta_i)\, H_z(d_i)\, H_x(a_i)\, H_x(\alpha_i)$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c\theta_i = \cos\theta_i$ $\qquad$ $c\alpha_i = \cos\alpha_i$

$s\theta_i = \sin\theta_i$ $\qquad$ $s\alpha_i = \sin\alpha_i$

To Represent 3D motion: 3 positions (x, y, z)

3 translation $(\theta_x, \theta_y, \theta_z)$

$\downarrow$

we need 6 numbers

But D-H convention uses only 4 numbers: $\theta_i, d_i, a_i, \alpha_i$

This is possible because DH uses a special way to define the axis of the links

These 2 special things are:

1. axis of $\underline{x_i}$ is perpendicular to $\underline{z_{i-1}}$

2. axis of x_i intersects with z_{i-1}

These 2 constraints help to get rid of 2 parameters. Hence DH uses only 4 parameters.

Algorithm for using DH for forward kinematics There are three steps.

1. **Assign coordinate frames:**
   (a) Assign z_i along the axis of actuation for each link, where $i = 0, 1, 2, ...(n-1)$.
   (b) Assign the base frame $o_0 - x_0 - y_0 - z_0$. The z_0 has already been assigned. Assign x_0 arbitrarily. Assign y_0 based on x_0 and z_0 using right hand rule.
   (c) Now assign coordinate frames $o_i - x_i - y_i - z_i$ for $i = 1, 2, ..., n-1$. z_i is already attached in first step. Next we assign x_i using these rules.
      i. **z_{i-1} and z_i are not coplanar:** In this case, there is a unique shorted distance segment that is perpendicular to z_{i-1} and z_i. Choose this as x_i axis. The origin o_i is where x_i intersects z_i. The y_i is found from right hand rules.
      ii. **z_{i-1} and z_i parallel:** In this case, there infinitely many perpendiculars. Choose any of these perpendiculars for x_i. Furthermore, where x_i intersects z_i we draw the origin x_i. Finally, y_i is found from the right hand rule. To make equations simpler, choose x_i such that is passes through o_{i-1}. This will make $d_i = 0$. Also, since z_{i-1} is parallel to z_i, $\alpha_i = 0$.
      iii. **z_{i-1} and z_i intersect:** In this case, x_i is chosen to be normal to the plane formed by z_{i-1} and z_i. There will be two possible directions for x_i, one of them is chosen arbitrarily and o_i is obtained by the intersection of $z - i$ and x_i. Finally y_i is obtained from right hand rule. Also, since z_{i-1} intersects z_i, $a_i = 0$.
   (d) Finally we need to attach an end effector frame, $o_n - x_n - y_n - z_n$. Attach z_n to be the same direction as z_{n-1}. Now depending on the relation between z_n and z_{n-1}, attach frame x_n. Finally, attach y_n using the right hand rule.

DH algorithm (2 of 2)

2. **Generate a table for DH parameter:** Now generate the DH table as follows.

Link	a_i	α_i	d_i	θ_i
1				
2				
.				
.				
.				
n				

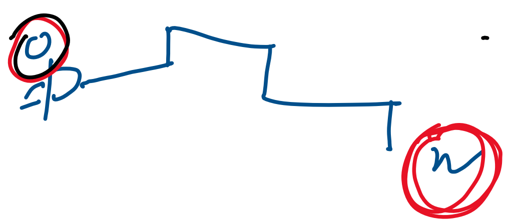
3. **Apply DH transformation to evaluate forward kinematics:** Finally, use the DH formulate to link two adjacent frames

$$\mathbf{H}_i^{i-1} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The position and orientation of the end-effector is found using the formula

$$\mathbf{H}_n^0 = \mathbf{H}_1^0\mathbf{H}_2^1\mathbf{H}_3^2 ... \mathbf{H}_n^{n-1} = \begin{bmatrix} \mathbf{R}_n^0 & \mathbf{d}_n^0 \\ \mathbf{0} & 1 \end{bmatrix}$$

The position of the end-effector is $\mathbf{d}_n^0$ and the orientation is $\mathbf{R}_n^0$. From $\mathbf{R}_n^0$, we can recover the Euler angles for the end-effector frame.

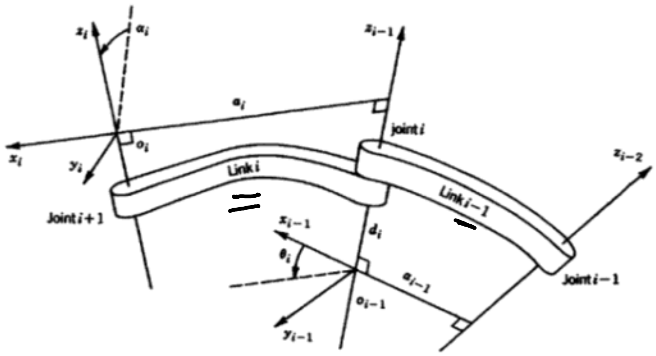


R_n^0 - orientation of n-th axis w.r.t. 0

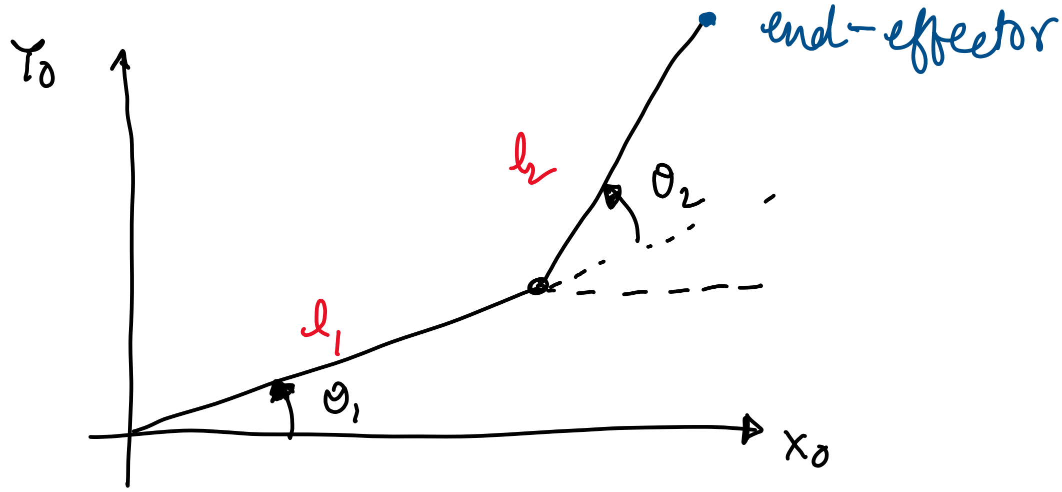
d_n^0 - position of n^{th} axis w.r.t. 0.

DH figure

1. a_i is the distance between z_i and z_{i-1} along x_i.
2. α_i is the angle between z_i and z_{i-1} along x_i.
3. d_i is the distance between x_{i-1} and x_i along z_{i-1}.
4. θ_i is the angle between x_{i-1} and x_i along z_{i-1}.



EXAMPLE 1



Compute the position and orientation of the end-effector

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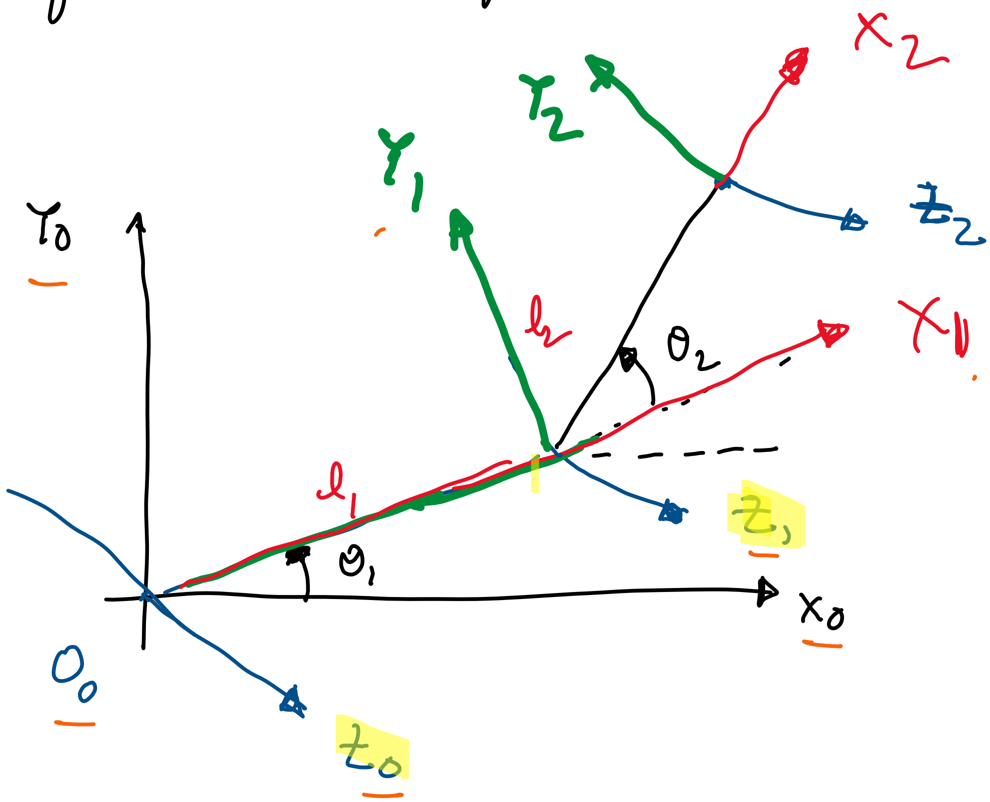
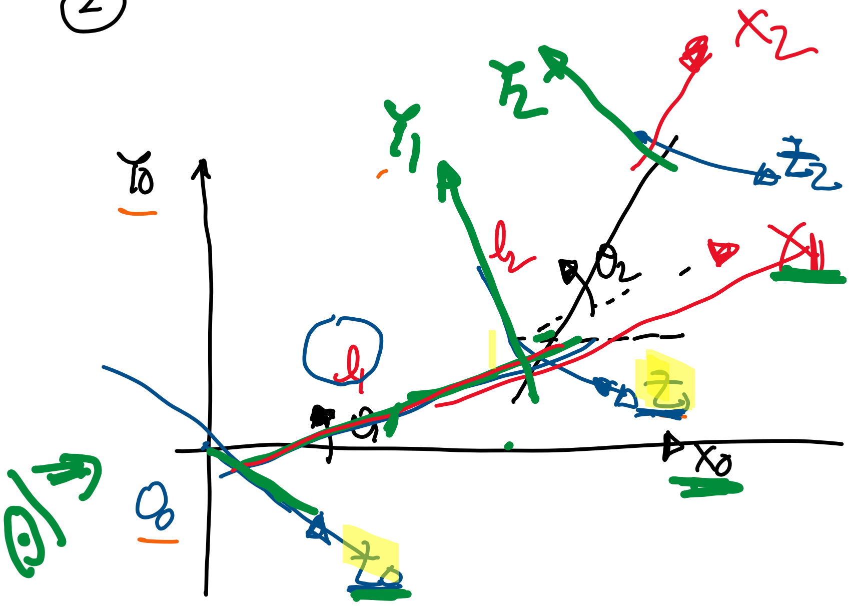
(1) Assign co-ordinate frames

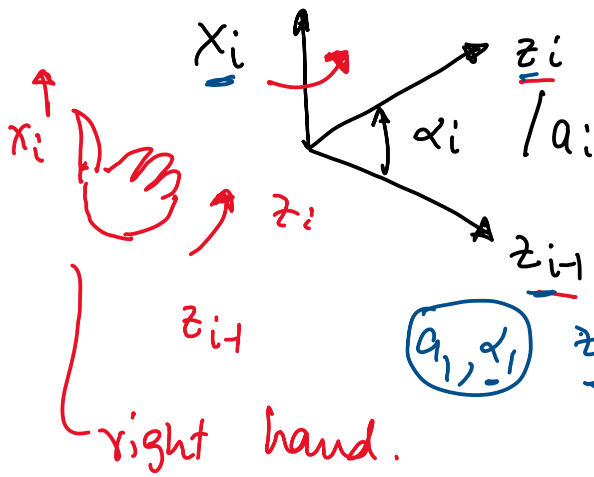


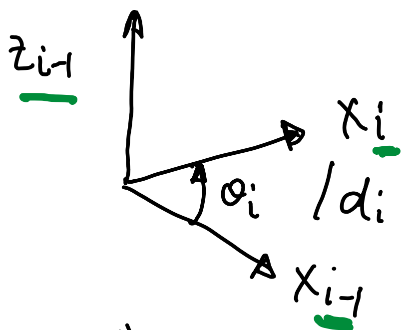
2



DH Table

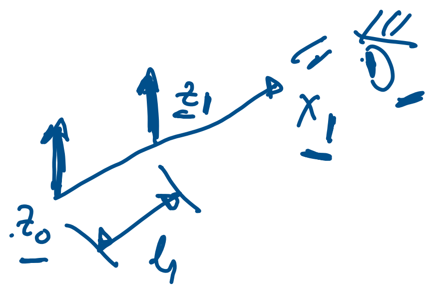
Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2				

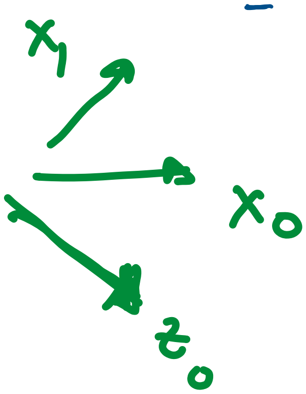


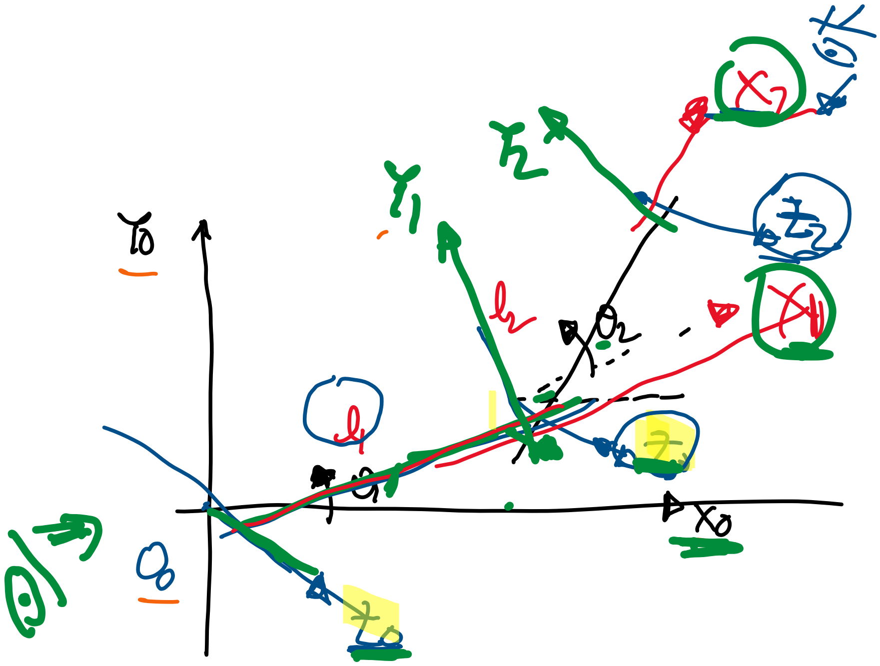


a_1, α_1 $z_0 - z_1 \parallel x_1$

d_1, θ_1 $x_0 - x_1 \mid z_0$

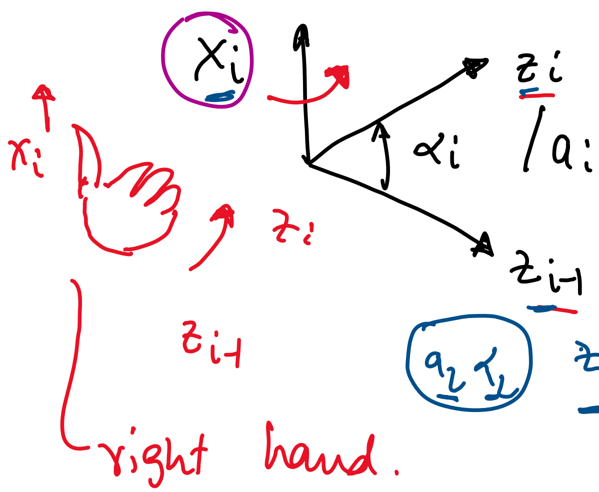


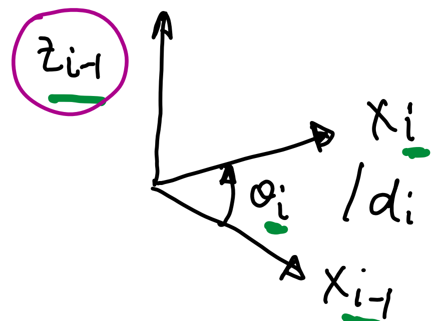




DH Table

Link	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2





a_2 α_2 $z_1 - z_2 \parallel x_2$

d_2 θ_2 $x_1 - x_2 | z_1$

③ Use DH transformation

$$H_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} c_1 = \cos\theta_1 \\ s_1 = \sin\theta_1 \end{array}$$

From the first line of the DH table

	a_1	α_1	d_1	θ_1
Link 1	l_1	0	0	θ_1

$$H_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} c_2 = \cos\theta_2 \\ s_2 = \sin\theta_2 \end{array}$$

From the 2nd line of the DH table

	a_2	α_2	d_2	θ_2
Link 2	l_2	0	0	θ_2

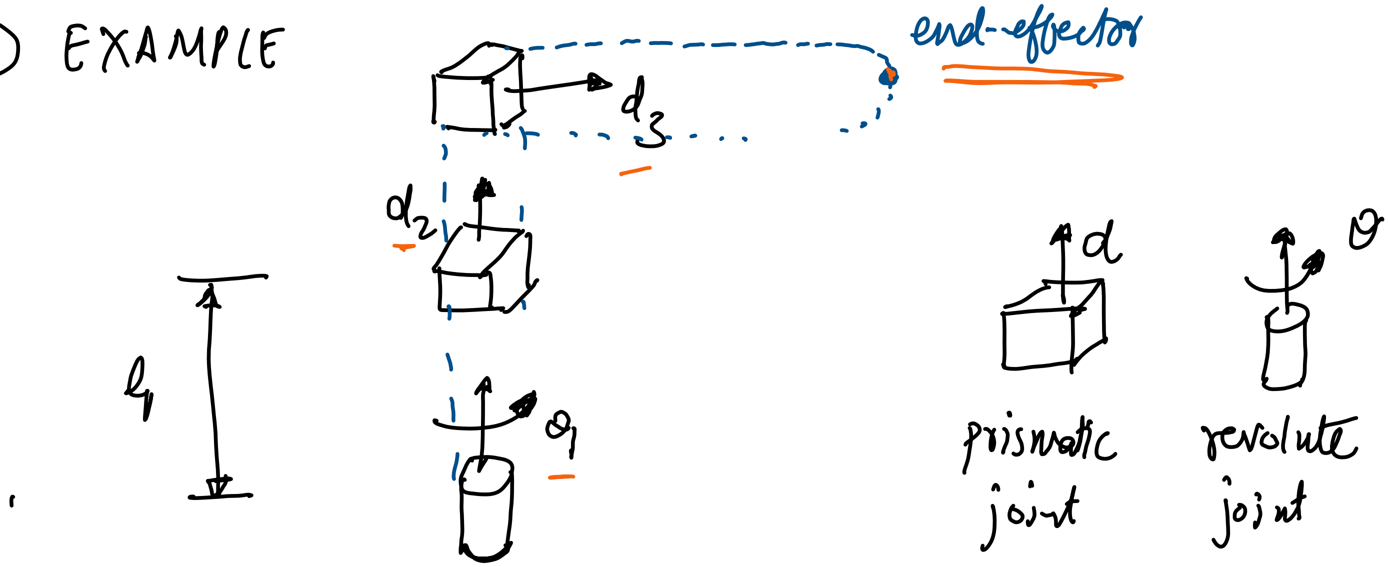
position of the end-effector

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c_{12} = \cos(\theta_1 + \theta_2)$; $s_{12} = \sin(\theta_1 + \theta_2)$

orientation of the end effector

(2) EXAMPLE



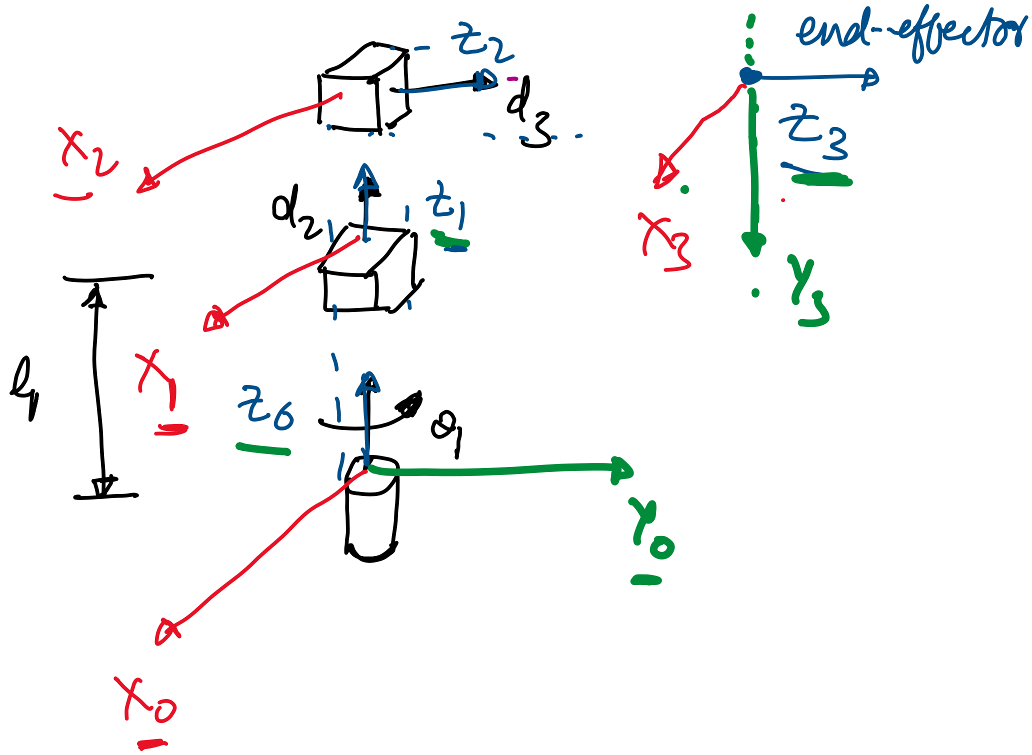
Links are NOT shown
Only the joints are shown.

✓ 1 Revolute (1R)
✓ 2 prismatic (2P)

1R-2P manipulator

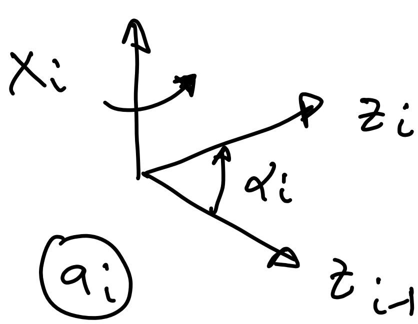
Compute the position and orientation of the end-effector using DH convention

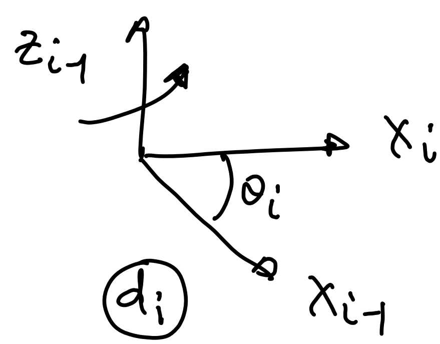
① Assign co-ordinate frames

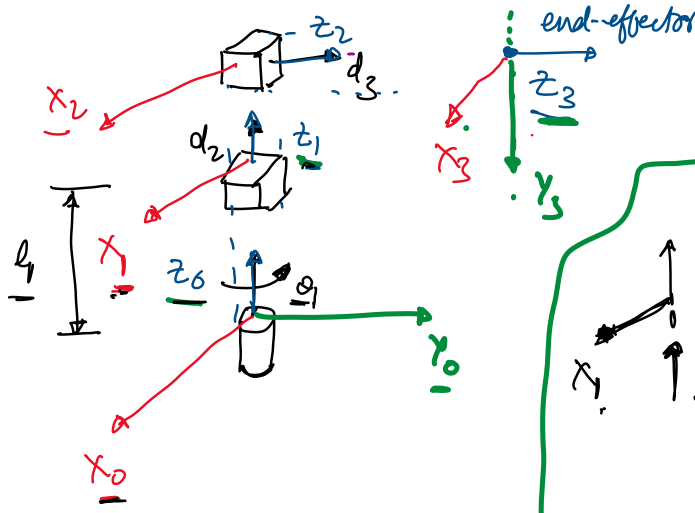


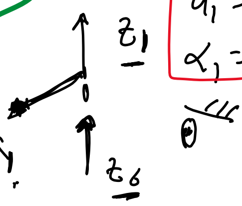
② DH table

Link	a_i	α_i	d_i	θ_i
1	0	0	l_1	θ_1
2	0	270 | -90	d_2	0
3	0	0	d_3	0



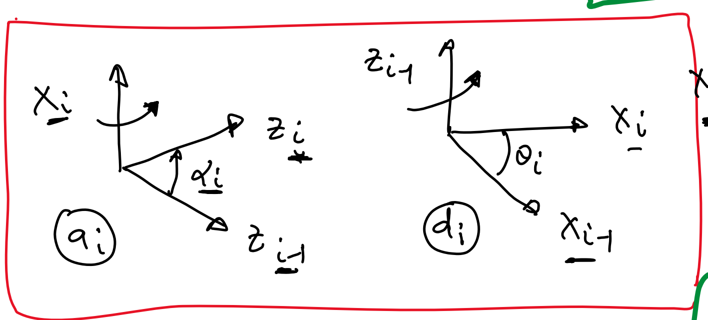


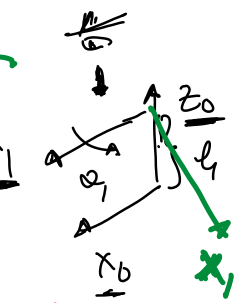




$a_1 = 0$

$\alpha_1 = 0$



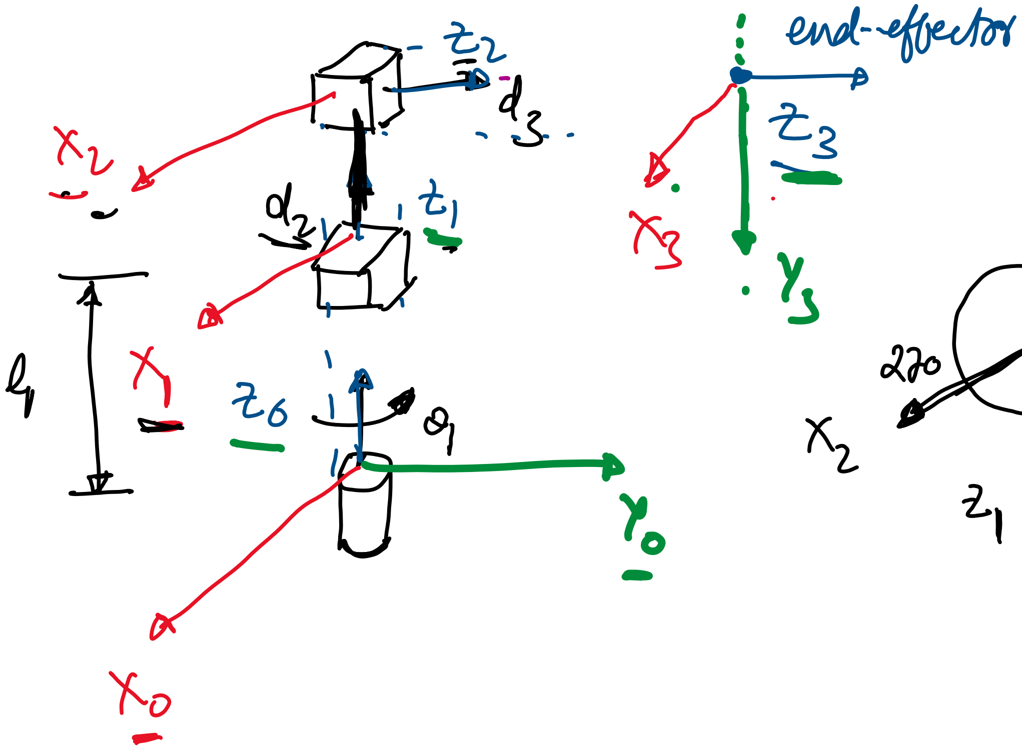


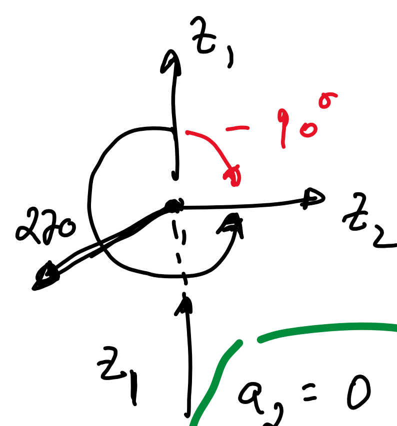
$d_1 = l_1$

$\theta_1 = \theta_1$

$i=1$ $\quad a_1, \alpha_1 \quad z_0, z_1, x_1$

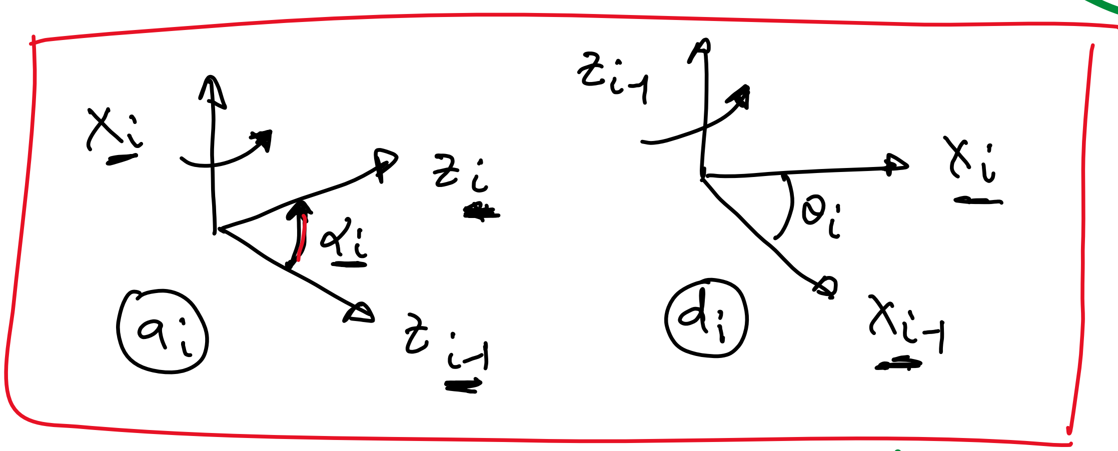
$d_1, \theta_1 \quad x_0, x_1, z_0$





$a_2 = 0$

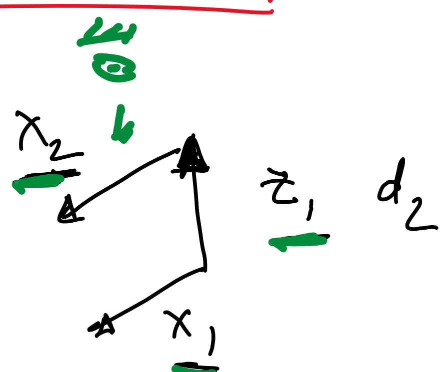
$\alpha_2 = 270°$ / $-90°$



$i = 2$

z_1, z_2, x_2

x_1, x_2, z_1

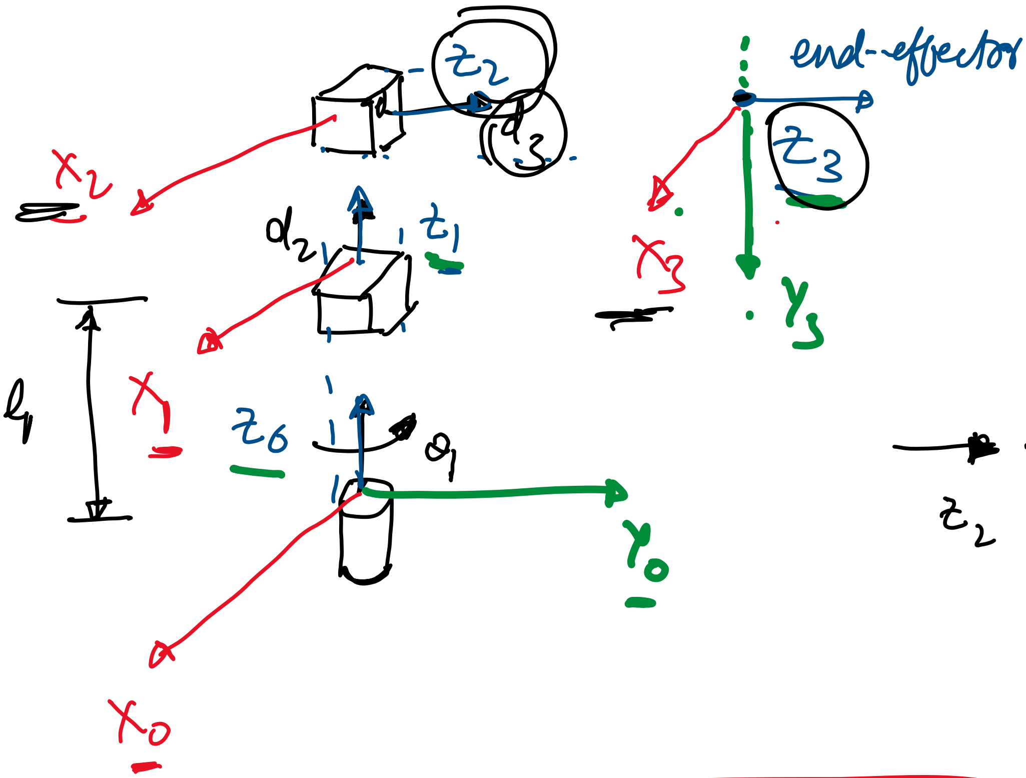


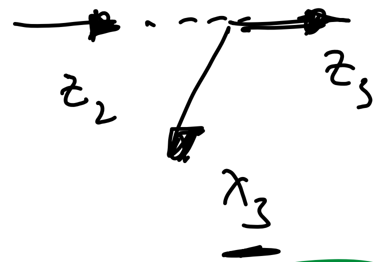
$d_2 = d_2$

$\theta_2 = 0$

$d_2 = d_2$

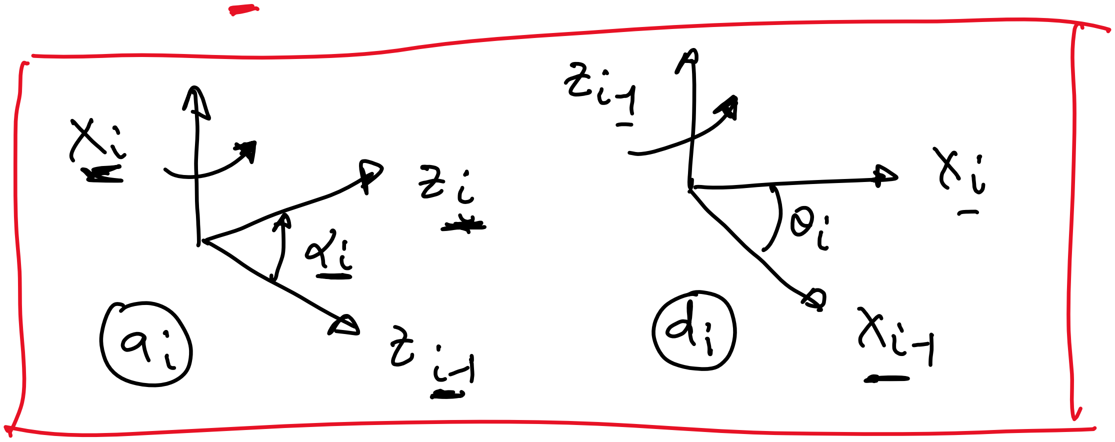
$\theta_2 = 0$





$a_3 = 0$

$\alpha_3 = 0$



$i = 3$ $\quad a_3, \alpha_3 \quad z_2, z_3, x_3$

$d_3, \theta_3 \quad x_2, x_3, z_2$

$d_3 = d_3$

$\theta_3 = 0$

③ Compute H_i^{i-1} using the D-H formula.

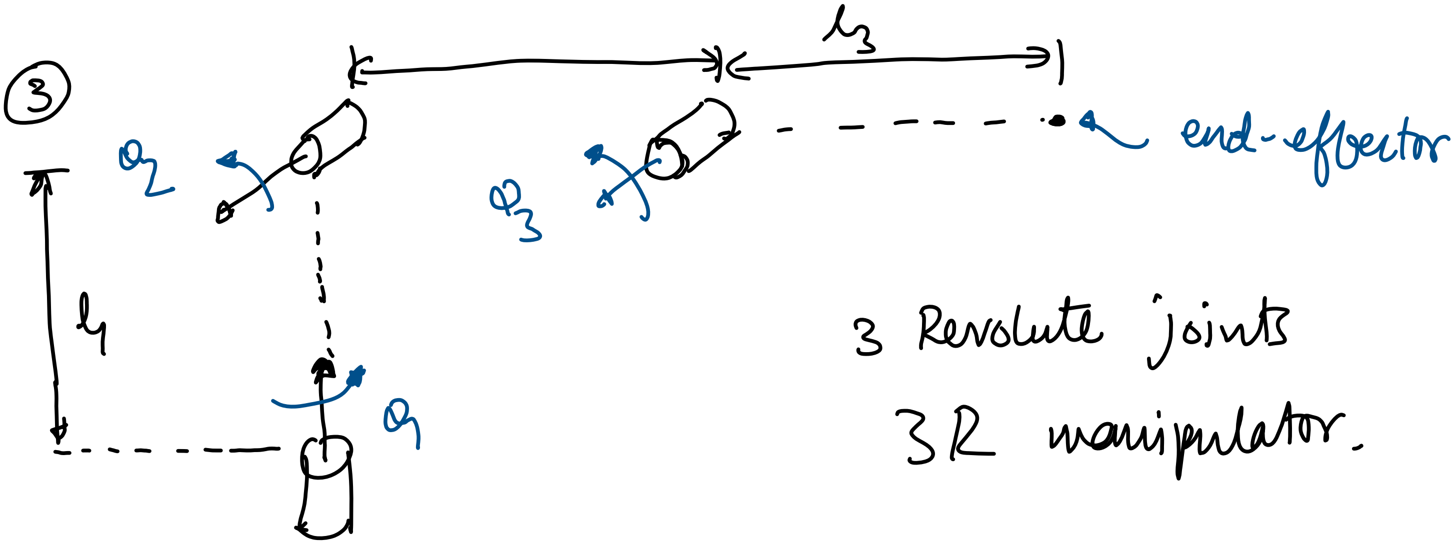
H_1^0, H_2^1, H_3^2 ✓

$$H_3^0 = H_1^0 H_2^1 H_3^2 = \begin{bmatrix} (R_3^0) & (d_3^0) \\ 0 & 1 \end{bmatrix}$$

position of the end-effector

orientation of the end-effector

③



3 Revolute joints

3R manipulator.

Compute the position and orientation of the end-effector

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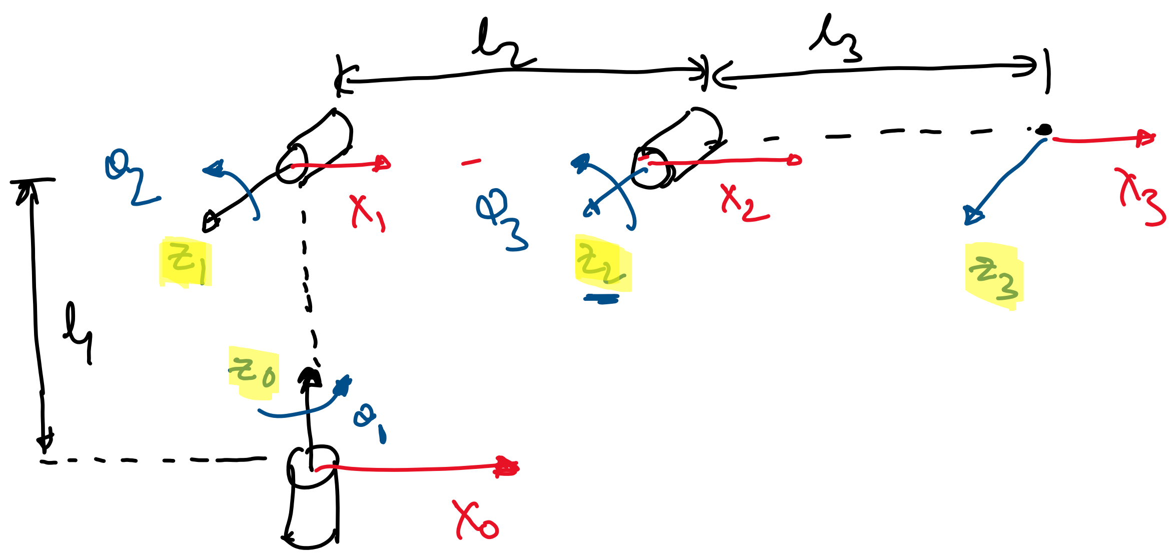
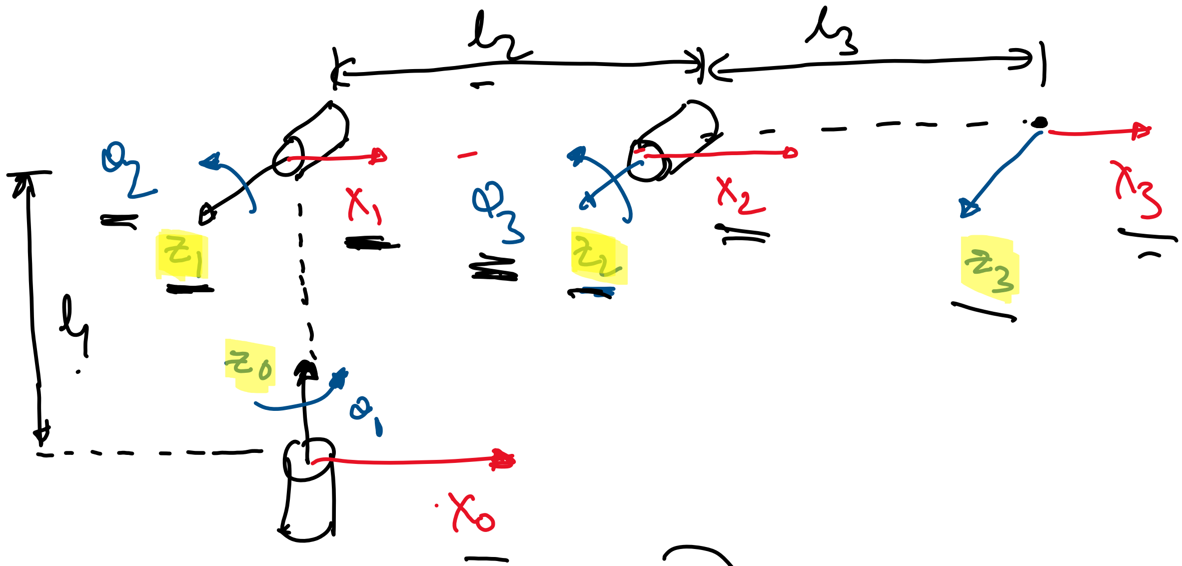
①



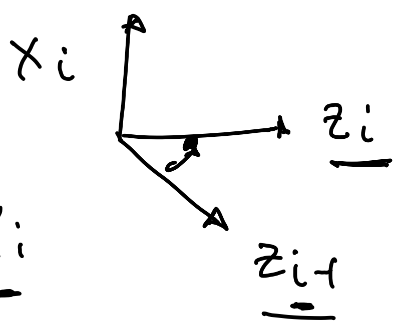
$x_1 \perp z_0, z_1$



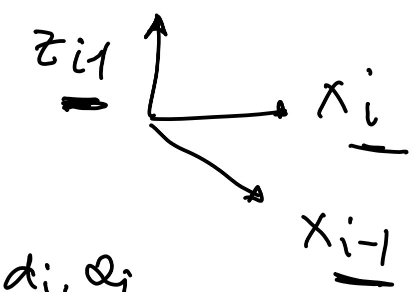
2

z_1

Link	a_i	α_i	d_i	θ_i
1	0	90°	l_1	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3



a_i, α_i



d_i, θ_i

(3) $H_i^{i-1} = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}_{4\times4}$

$\underline{i=1}$ $\quad H_1^0 = [\quad]_{4\times4}$

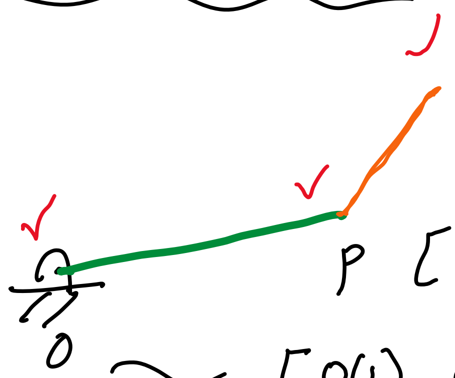
$i=2$ $\quad H_2^1 = [\quad]_{4\times4}$

$i=3$ $\quad H_3^2 = [\quad]_{4\times4}$

$$H_3^0 = H_1^0 H_2^1 H_3^2 = \begin{bmatrix} \underbrace{[R_3^0]}_{\substack{3\times3 \\ \text{orientation}}} & \underbrace{[O_3^0]}_{\substack{3\times1 \\ \text{position}}} \\ 0 & 1 \end{bmatrix}$$

Animation / Visualization

2D



Q $[Q(1) \quad Q(2)]$ with $Q(1)$ = x, $Q(2)$ = y

P $[P(1) \quad P(2)]$ with $P(1)$ = x, $P(2)$ = y

O ~ $[O(1) \quad O(2)]$ with $O(1)$ = x, $O(2)$ = y

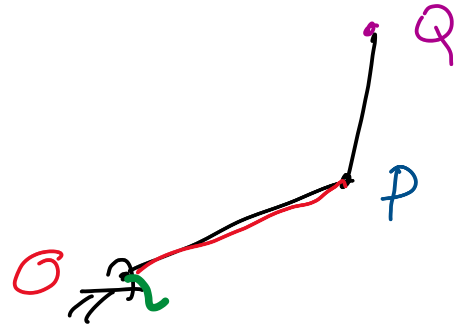
we compute $(O[1], O[2])$, $(P(1), P(2))$, $[Q(1), Q(2)]$

using homogeneous transformation in 2D

$$\text{line}\left(\underbrace{[O(1) \quad P(1)]}_{x}, \underbrace{[O(2) \quad P(2)]}_{y}\right)$$

$$\text{line}\left([P(1), Q(1)], [P(2), Q(2)]\right)$$

3D



$H_1^0, \quad H_2^1, \quad H_3^2$ ✓

$H_1^0, \quad H_2^0 = H_1^0 H_2^1, \quad H_3^0 = H_1^0 H_2^1 H_3^2$

$$\begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} R_2^0 & O_2^0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} R_3^0 & O_3^0 \\ 0 & 1 \end{bmatrix}$$

$O(1), O(2), O(3)$

$P(1), P(2), P(3)$

$Q(1), Q(2), Q(3)$

$$\text{line}\left(\underbrace{(O(1)\ P(1))}_{x}, \underbrace{(O(2)\ P(2))}_{y}, \underbrace{(O(3), P(3))}_{z}\right)$$

$$\text{line}\left(\underbrace{(P(1), Q(1))}_{x}, \underbrace{(P(2), Q(2))}_{y}, \underbrace{(P(3), Q(3))}_{z}\right)$$