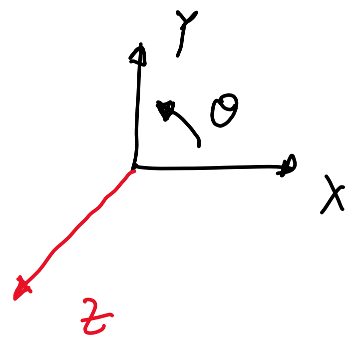
3 D Angular velocity

In 2D: $\vec{\omega}_z = \dot{\theta}\,\hat{k}$ ✓



$\hat{k}$ unit vector along z-axis

In 2D : $\vec{v} = \vec{\omega}_z \times \vec{r} = J_q\,\dot{q}$ ✓

$\vec{v}$ – linear velocity
$\vec{r}$ – position vector

$$\vec{v} = \vec{\omega} \times \vec{r} = (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \times (r_x \hat{i} + r_y \hat{j} + r_z \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} \omega_y & \omega_z \\ r_y & r_z \end{vmatrix} - \hat{j} \begin{vmatrix} \omega_x & \omega_z \\ r_x & r_z \end{vmatrix} + \hat{k} \begin{vmatrix} \omega_x & \omega_y \\ r_x & r_y \end{vmatrix}$$

$$= \hat{i}\,(\omega_y r_z - \omega_z r_y) - \hat{j}\,(\omega_x r_z - \omega_z r_x) + \hat{k}\,(\omega_x r_y - \omega_y r_x) = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix}$$

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skew - symmetric matrix (S)

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Definition

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$S(a) + S^T(a) = 0_{3\times3}$$

Properties

We will prove this

① $\vec{a} \times \vec{b} = S(a)\, b$ || cross-product = (matrix) vector

② $R\, S(a) R^T = S(Ra)$ R = rotation matrix

$\vec{a} = \vec{\omega}$ & $\vec{b} = \vec{r}$

Then we need to show that

$$\vec{\omega} \times \vec{r} = S(\omega)\, r$$

$$\vec{\omega} \times \vec{r} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} \quad - (1)$$

$$S(\omega)\, r = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$= \begin{bmatrix} -\omega_z r_y + \omega_y r_z \\ \omega_z r_x - \omega_x r_z \\ -\omega_y r_x + \omega_x r_y \end{bmatrix} \quad - (2)$$

$(1) = (2)$

$$\vec{\omega} \times \vec{r} = S(\omega)\, r \quad \checkmark$$

In 3D $\quad \vec{\omega} \neq \dot{\phi}\hat{i} + \dot{\theta}\hat{j} + \dot{\psi}\hat{k}$

Does not hold in 3D because rotations are not commutative

Goal: Derive an expression for $\vec{\omega}$ in 3D.
This should be a function of Euler angles (ϕ, θ, ψ) and angular rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$

We know that $\quad R^T R = I \quad \Rightarrow \quad RR^T = I$

Differentiate with respect to time

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$\dot{R}^T R + \left[\left(R^T \dot{R} \right)^T \right]^T = 0$$

$$\dot{R}^T R + \left[\dot{R}^T R \right]^T = 0$$

used $(AB)^T = B^T A^T$

$$\dot{R}^T R + \left(\dot{R}^T R \right)^T = 0$$

$$S(a) + S^T(a) = 0$$

skew symmetric

vector

$$\dot{R}^T R = S(a)$$

$$\dot{R}^T R = S(a)$$

Post-multiply with R^T

$$\dot{R}^T \underbrace{R R^T}_{I} = S(a) R^T$$

$$\dot{R}^T = S(a) R^T$$

Replace R^T as R

$$\underline{\dot{R}} = S(a) R \quad \text{— (I)}$$

I still dont have an expression for $\vec{w}$?

what is a?

We will show that $\underline{\vec{a}}$ is $\underline{\vec{w}}$

$$r = R r^b$$

r – position in world frame
r^b – position in body frame (from Lec 16)

$$\underline{r = R r^b} \qquad \text{— } II$$

Differentiate with respect to time

$$\dot{r} = \dot{R} r^b + R \dot{r}^b$$

$$\dot{r} = \underline{\dot{R}} r^b$$

But $\dot{R} = S(a) R$ from I

$$\dot{r} = \underline{S(a) R \; r^b}$$

$$\dot{r} = \underline{S(a)} r \quad \checkmark \qquad \text{from } II$$

we know that $\underline{\vec{a} \times \vec{b}} = \underline{S(a) b}$

$$\dot{\vec{r}} = \underline{\vec{a} \times \vec{r}} \quad \checkmark \qquad \text{— } III$$

But we know that $\vec{v} = \dot{\vec{r}} = \underline{\vec{\omega} \times \vec{r}}$ — IV

From III and IV $\vec{a} = \vec{\omega}$ ✓

$$\boxed{\dot{R} = S(\omega) R \qquad S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}$$

How are $\omega_x, \omega_y, \omega_z$ related to $\dot\phi, \dot\theta, \dot\psi$?

$$S(\omega) = \dot R R^T \checkmark$$

We chose 3-2-1 $\quad R = R_z R_y R_x$

$$= R_z(\psi) R_y(\theta) R_x(\phi)$$

$$S(\omega) = \dot{\overline{(R_z R_y R_x)}} (R_z R_y R_x)^T$$

$$= \dot{\overline{(R_z R_y R_x)}} (R_x^T R_y^T R_z^T)$$

$$\left[(ABC)^T = C^T B^T A^T \right]$$

$$S(\omega) = \Big(\underbrace{\dot R_z R_y R_x}_{①} + \underbrace{R_z \dot R_y R_x}_{②} + \underbrace{R_z R_y \dot R_x}_{③} \Big) \underbrace{R_x^T R_y^T R_z^T}_{Ⓘ}$$

$$① \quad \dot R_z R_y \underbrace{R_x R_x^T}_{I} R_y^T R_z^T = \dot R_z \underbrace{R_y R_y^T}_{I} R_z^T$$

$$= \dot R_z R_z^T = S(\dot\psi \hat k)$$

see this

$$S(\omega_z) = \dot R_z R_z^T = S(\dot\psi \hat k)$$

$$② \; R_z \dot{R}_y \underbrace{R_x R_x^T}_{=I} R_y^T R_z^T = R_z \dot{R}_y R_y^T R_z^T$$

But $S(\omega_y) = \dot{R}_y R_y^T = S(\dot{\theta}\hat{j})$

$$= R_z \, S(\dot{\theta}\hat{j}) \, R_z^T$$

But $R \, S(a) \, R^T = S(Ra)$

$$= S(R_z \dot{\theta}\hat{j})$$

$$③ \; R_z R_y \underbrace{\dot{R}_x R_x^T}_{S(\omega_x)} R_y^T R_z^T = R_z R_y \, S(\omega_x) R_y^T R_z^T$$

$$= R_z R_y \, S(\dot{\phi}\hat{i}) R_y^T R_z^T$$

$$= R_z R_y \, S(\dot{\phi}\hat{i}) (R_z R_y)^T$$

$(AB)^T = B^T A^T$

$$= S(R_z R_y \dot{\phi}\hat{i})$$

From (I)

$$S(\omega) = S(\dot{\psi}\hat{k}) + S(R_z\dot{\theta}\hat{j}) + S(R_zR_y\dot{\phi}\hat{\imath})$$

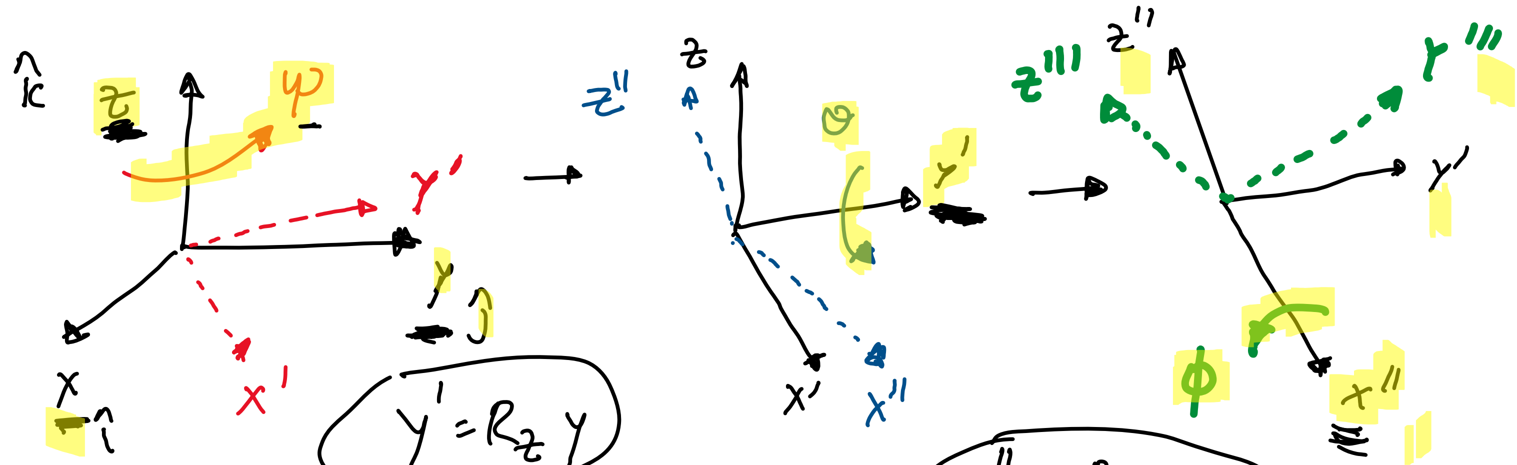
$$S(\omega) = S(\dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_zR_y\dot{\phi}\hat{\imath})$$

$$\boxed{\vec{\omega} = \dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_zR_y\dot{\phi}\hat{\imath}}$$ ✓

This is for 3-2-1 rotations.
$\psi - \theta - \phi$

3D angular velocity for 3-2-1 rotation.

Intuitive method to derive $\vec{\omega}$



$$y' = R_{\underline{z}}\, y$$

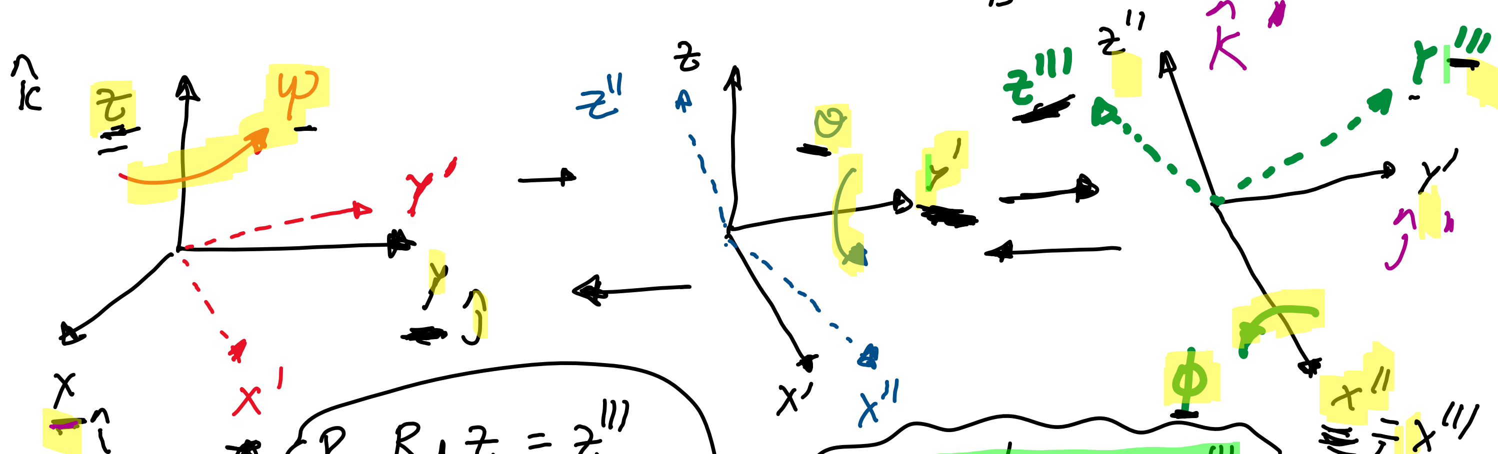
$$x'' = R_z R_y x$$

$$\vec{\omega} = \dot{\psi}\, \underline{z} + \dot{\theta}\, \underline{y}' + \dot{\phi}\, \underline{x}''$$

$$= \dot{\psi}\, \hat{k} + \dot{\theta} \underbrace{\left(R_z \hat{j}\right)}_{y'} + \dot{\phi} \underbrace{R_z R_y \hat{i}}_{x''}$$

$$\boxed{\vec{\omega} = \dot{\psi}\, \hat{k} + \dot{\theta}\, R_z \hat{j} + \dot{\phi}\, R_z R_y \hat{i}}$$

Intuitive method to derive $\vec{\omega}_b$ → body frame



$$R_\theta R_\phi z = z'''$$

$$z = R_\phi^T R_\theta^T z'''$$

$$R_\phi y' = y'''$$

$$y' = R_\phi^T y'''$$

$$y' = R_\phi^T \hat{j}'$$

$x'' = x'''$, $\hat{i}'$

$\hat{i}', \hat{j}', \hat{k}'$ are in body frame

$$\vec{\omega}_B = \dot{\phi}\, \underline{x''} + \dot{\theta}\, \underline{y'} + \dot{\psi}\, z$$

$$\boxed{\begin{aligned} \vec{\omega}_B &= \dot{\phi}\, \hat{i}' + \dot{\theta}\, R_\phi^T \hat{j}' + \dot{\psi}\, R_\phi^T R_\theta^T \hat{k}' \\ &= \dot{\phi}\, \hat{i}' + \dot{\theta}\, R_x^T \hat{j}' + \dot{\psi}\, R_x^T R_y^T \hat{k}' \end{aligned}}$$

Correction.

Writing these in matrix form

$$\omega = \begin{bmatrix} \cos\psi \cos\theta & -\sin\psi & 0 \\ \cos\theta \sin\psi & \cos\psi & 0 \\ -\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\omega_B = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\phi \cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\omega = A \dot\Theta$$

$$\omega_B = A_B \dot\Theta$$

$$\dot\Theta = \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\Theta = A^{-1} \omega$$

$$\Theta = A_B^{-1} \omega_B$$

$$\det(A) = \det(A_B) = \cos\phi$$

A^{-1} or A_B^{-1} will have $\frac{1}{\cos\phi}$.

not defined when $\phi = 90^\circ$. singularity

AVOID singularity use QUATERNIONS