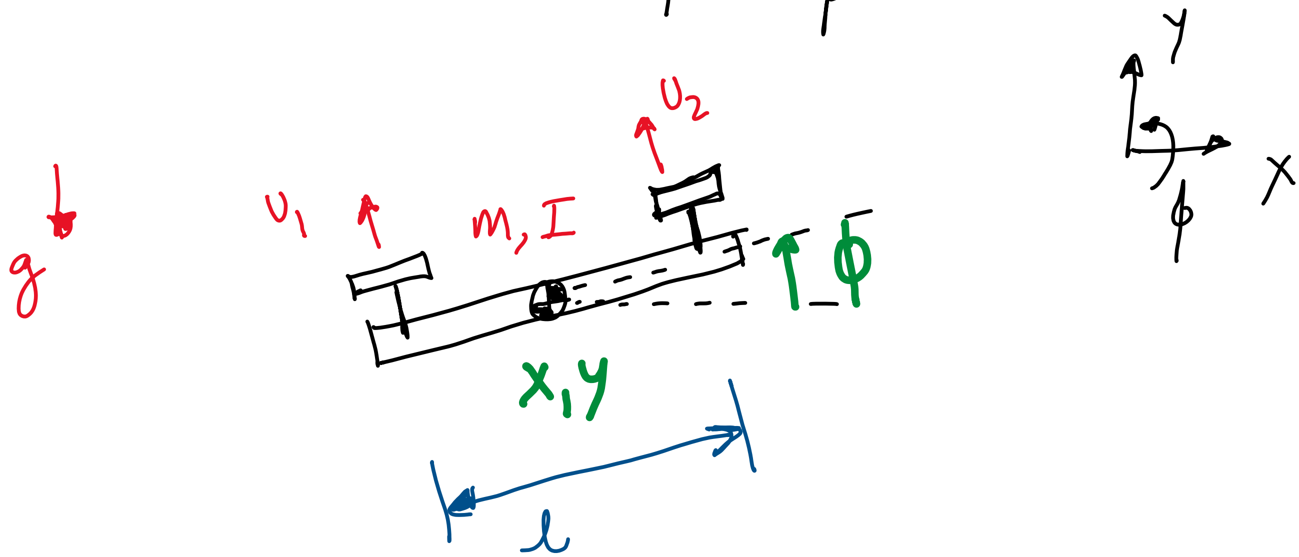
Bicopter

2D version of a quadcopter



m, I - mass, inertia
g, l - gravity, distance between the propellors
u_1, u_2 - thrust forces
x, y, ϕ - degrees of freedom

Equations of motion

Euler-lagrange method

① Get the positions / velocities of the center of mass.

x, y, ϕ - positions

$\dot{x}, \dot{y}, \dot{\phi}$ - velocities

② Get the kinetic / Potential energy

$T = 0.5\, m(\dot{x}^2 + \dot{y}^2) + 0.5\, I\dot{\phi}^2$ ✓

$V = mgy$

$T = 0.5\, m(\dot{x}^2 + \dot{y}^2) + 0.5\, I \dot{\varphi}^2$ ✓

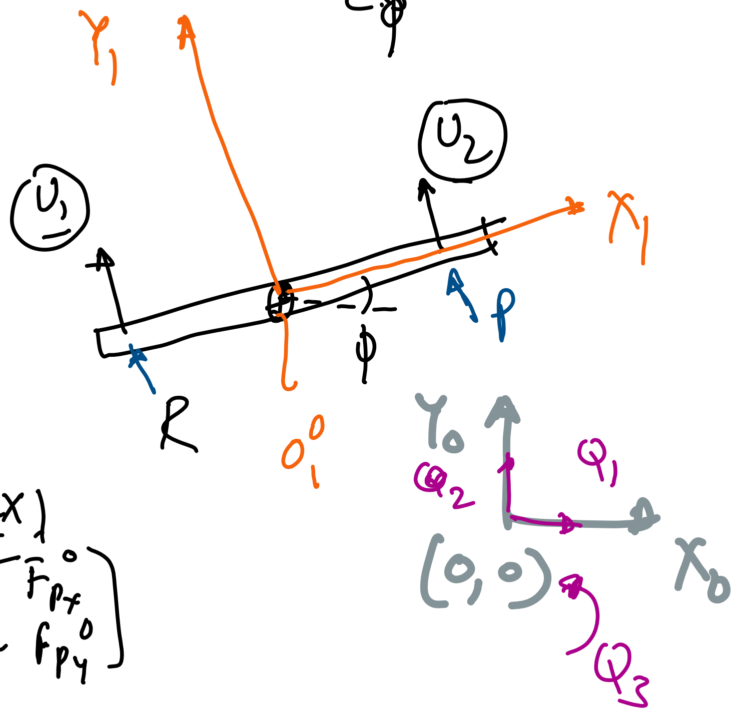
$V = mgy$ ✓

$L = T - V$

In Euler-Lagrange, we need to compute generalized forces Q_j

Recap: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \boxed{Q_j}$$ — $\left.\begin{matrix} F_x \\ F_y \\ \tau_\phi \end{matrix}\right\}$

③ Compute Q_j



$$\checkmark\ Q_j = J_P^T F_P^0 + J_R^T F_R^0$$

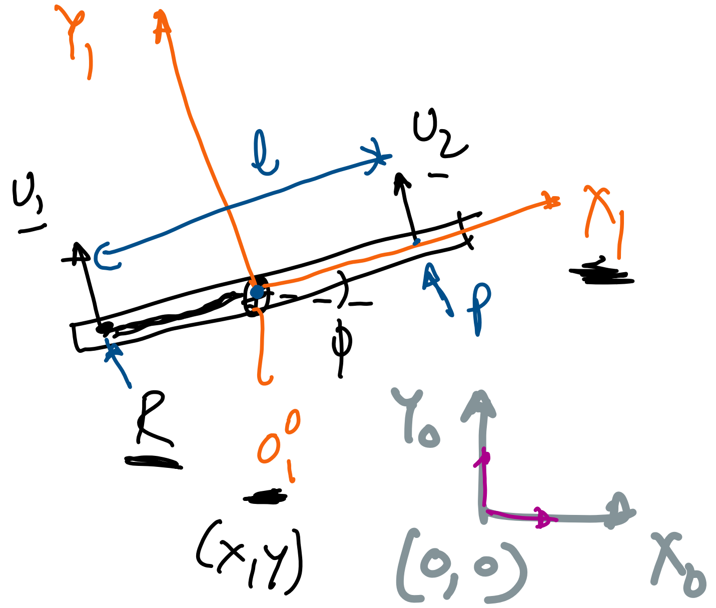
$J^0 = \frac{\partial \cancel{X}_P}{\partial q}$ $\quad J^0 = \frac{\partial X_R}{\partial q}$ $\quad$ 3x2 $\quad$ 2x1 $\begin{bmatrix} F^0_{P_x} \\ F^0_{P_y} \end{bmatrix}$

$q = (x, y, \phi)$

$X_P^0 = [\,x_P^0 \quad y_P^0\,]$

$X_R^0 = [\,x_R^0 \quad y_R^0\,]$

$$H_1^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}$$



$$H_1^0 = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$p^0 = H_1^0 \, p^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x + 0.5\,l\cos\phi \\ y + 0.5\,l\sin\phi \\ 1 \end{bmatrix} \begin{matrix} = x_p^0 \\ = y_p^0 \\ \end{matrix}$$

$$r^0 = H_1^0 \, r^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x - 0.5\,l\cos\phi \\ y - 0.5\,l\sin\phi \\ 1 \end{bmatrix} \begin{matrix} = x_r^0 \\ = y_r^0 \\ \end{matrix}$$

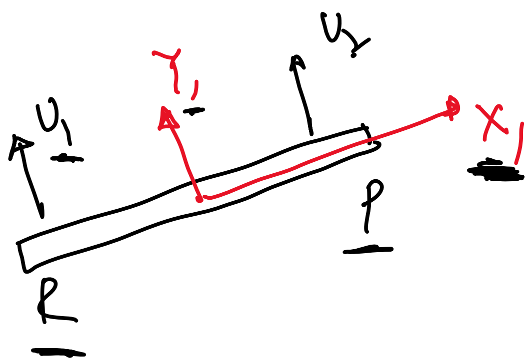
$$J_p = \frac{\partial X_p^0}{\partial q} = \begin{bmatrix} \frac{\partial x_p^0}{\partial x} & \frac{\partial x_p^0}{\partial y} & \frac{\partial x_p^0}{\partial \phi} \\ \frac{\partial y_p^0}{\partial x} & \frac{\partial y_p^0}{\partial y} & \frac{\partial y_p^0}{\partial \phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.5\,l\sin\phi \\ 0 & 1 & 0.5\,l\cos\phi \end{bmatrix}$$

$$J_R = \frac{\partial X_R^0}{\partial q} = \begin{bmatrix} \frac{\partial x_r^0}{\partial x} & \frac{\partial x_r^0}{\partial y} & \frac{\partial x_r^0}{\partial \phi} \\ \frac{\partial y_r^0}{\partial x} & \frac{\partial y_r^0}{\partial y} & \frac{\partial y_r^0}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0.5\,l \sin\phi \\ 0 & 1 & -0.5\,l \cos\phi \end{bmatrix}$$

$$F_P^0 = R_1^0 F_P^1$$

force in frame 1



$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} -u_2 \sin\phi \\ u_2 \cos\phi \end{bmatrix}$$

$$F_R^0 = R_1^0 F_R^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \end{bmatrix}$$

$$= \begin{bmatrix} -u_1 \sin\phi \\ u_1 \cos\phi \end{bmatrix}$$

$$Q_j = J_p^T F_p^o + J_R^T F_R^o$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{l}{2}\sin\phi & \frac{l}{2}\cos\phi \end{bmatrix} \begin{bmatrix} -u_2\sin\phi \\ u_2\cos\phi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{l}{2}\sin\phi & \frac{l}{2}\cos\phi \end{bmatrix} \begin{bmatrix} -u_1\sin\phi \\ u_1\cos\phi \end{bmatrix}$$

$$Q_j = \begin{bmatrix} -(u_1+u_2)\sin\phi \\ (u_1+u_2)\cos\phi \\ -(u_1-u_2)(0.5l) \end{bmatrix}$$

Forces in x-y

moment in ϕ direction

④ Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$\mathcal{L} = 0.5\, m(\dot{x}^2 + \dot{y}^2) + 0.5\, I\dot{\phi}^2 - mgy$$

$q_j = x$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_1$$

$$\frac{d}{dt}(m\dot{x}) - 0 = -(u_1+u_2)\sin\phi$$

$$m\ddot{x} = -(U_1+U_2)\sin\phi$$

$$\ddot{x} = \frac{-(U_1+U_2)\sin\phi}{m} \quad — ①$$

$q_j = y \qquad \mathcal{L} = 0.5m(\dot{x}^2+\dot{y}^2) + 0.5I\dot{\phi}^2 - mgy$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = Q_2$$

$$\frac{d}{dt}\left(m\dot{y}\right) - (-mg) = (U_1+U_2)\cos\phi$$

$$\ddot{y} = \frac{(U_1+U_2)\cos\phi}{m} - g \quad — ②$$

$q_j = \phi$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = Q_3$$

$$\frac{d}{dt}\left(I\dot{\phi}\right) - 0 = -(U_1-U_2)\,0.5l$$

$$\ddot{\phi} = \frac{-(U_1-U_2)(0.5l)}{I} \quad — ③$$

$U_1 + U_2 = U_s$

$U_2 - U_1 = U_d$

simplify control.

Bicopter Equations

$$\ddot{x} = -\frac{U_s}{m}\sin\phi \qquad U_s = U_1 + U_2$$

$$\ddot{y} = +\frac{U_s}{m}\cos\phi - g \qquad U_d = U_2 - U_1$$

$$\ddot{\phi} = \frac{0.5\,l}{I}\,U_d$$

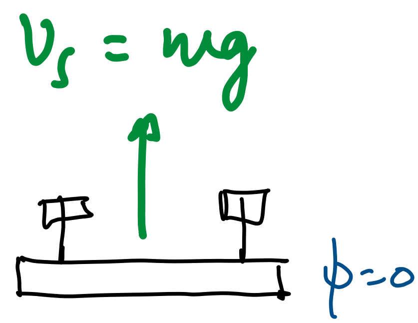
2 controls : U_1, U_2 or U_s, U_d

3 state variables: x, y, ϕ

2 controls $<$ 3 states

system is under-actuated.

Intuition



① $\phi = 0$ $\qquad \ddot{x} = 0$

$$\ddot{y} = \frac{U_s}{m} - g$$

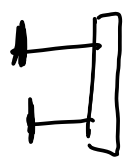
For bicopter to hover $\ddot{y} = 0 \Rightarrow U_s = mg$

$$\boxed{\begin{aligned} \ddot{x} &= -\frac{u_s}{m}\sin\phi \\ \ddot{y} &= \frac{u_s}{m}\cos\phi - g \\ \ddot{\phi} &= \frac{0.5\, l\, u_d}{I} \end{aligned}}$$

② $\phi = 90^\circ$

$$\left.\begin{aligned} \ddot{x} &= -\frac{u_s}{m} \Rightarrow \ddot{x} = 0 \\ & \qquad\qquad u_s = 0 \end{aligned}\right\}$$

$$\ddot{y} = 0 - g = -g$$



$\downarrow g$

quadcopter is uncontrollable when $\phi = 90^\circ$

③ $0 \leq \phi \leq \pi/2$ $\qquad \phi \simeq 0.1$

$$\ddot{x} = -\frac{u_s}{m}\sin\phi \approx -\frac{u_s\phi}{m}$$

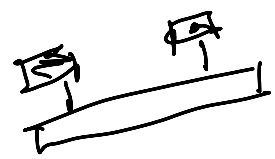
$$\ddot{y} = \frac{u_s}{m}\cos\phi - g \approx \frac{u_s}{m}(1) - g$$

$\cos\phi \approx 1$

ϕ is close to zero

$$\ddot{y} \approx 0 \qquad u_s = mg$$

$$\ddot{x} \approx -\frac{mg}{m}\phi \approx -g\phi$$

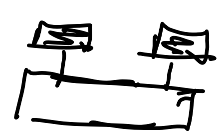


$\phi \approx 0.1$

moves left

④ $-\pi/2 \leq \phi \leq 0$

$$\ddot{x} \approx g\phi \qquad \phi \approx -0.1$$

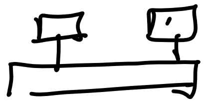


$$\ddot{y} \approx 0 \quad \text{when } u_s = mg$$

moves to the right

Intuitive control of bicopter

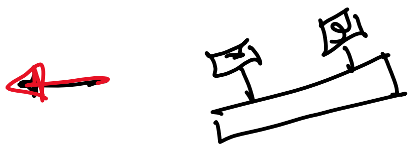
① Hover



$u_s \approx mg$

$\phi \approx 0$

② Move left



$\phi > 0$

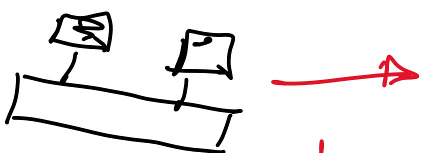
$u_s \approx mg$

Use u_d to chang

$\phi > 0$

$$\ddot{\phi} = u_d \left(\frac{0.5 l}{I} \right)$$

③ Move right



$\phi < 0$

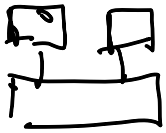
$u_s \approx mg$

Use u_d to change

$\phi < 0$

$$\ddot{\phi} = u_d \left(\frac{0.5 l}{I} \right)$$

④ Go up / down



$\phi \approx 0$

$u_s > mg$ $\uparrow$

$u_s < mg$ $\downarrow$

Feedback linearization to track a trajectory

x_{ref}, y_{ref}

$\dot{x}_{ref}, \dot{y}_{ref}$

$\ddot{x}_{ref}, \ddot{y}_{ref}$

$$\phi_{ref} = -\frac{1}{g}\left(\ddot{x}_{ref} + k_{p_x}(x_{ref} - x) + k_{dx}(\dot{x}_{ref} - \dot{x})\right)$$

$$\boxed{u_s} = mg + \delta u_s$$

$$\delta u_s = m\left(\ddot{y}_{ref} + k_{py}(y_{ref} - y) + k_{dy}(\dot{y}_{ref} - \dot{y})\right)$$

$$\boxed{u_d} = \delta u_d$$

$$\delta u_d = -k_{d\phi}\dot{\phi} - k_{p\phi}(\phi_{ref} - \phi)$$