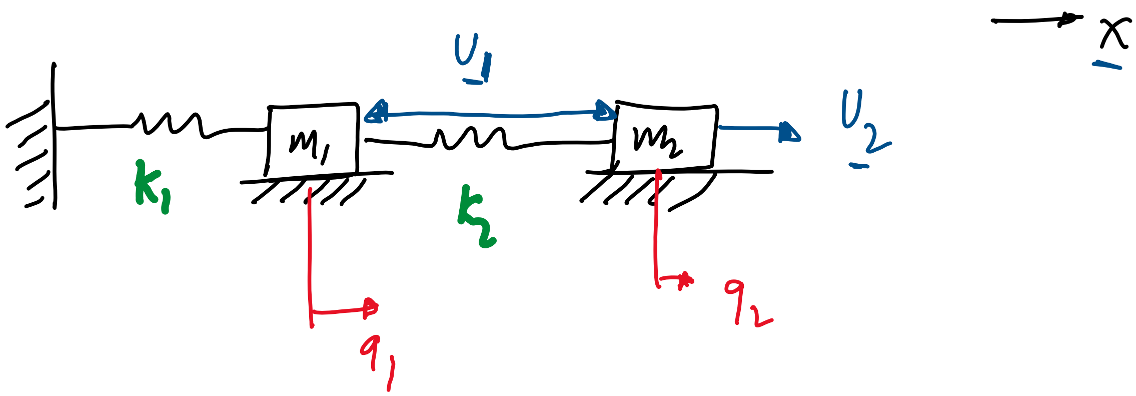
Linear control



Equations of motion

$$T = 0.5 \; m_1 \dot{q}_1^2 + 0.5 \; m_2 \dot{q}_2^2$$

$$V = 0.5 \; k_1 q_1^2 + 0.5 \; k_2 (q_1 - q_2)^2$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = 0.5 m_1 \dot{q}_1^2 + 0.5 m_2 \dot{q}_2^2 \quad 0.5 k_1 q_1^2 - 0.5 k_2 (q_1 - q_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) \quad \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

For q_1

$$\frac{d}{dt}\left(0.5 m_1 (2\dot{q}_1) \right) + \left(0.5 k_1 (2q_1) + 0.5 k_2 (2)(q_1 - q_2) \right) = -U_1$$

$$\rightarrow \quad m_1 \ddot{q}_1 + k_1 q_1 + k_2 (q_1 - q_2) = -U_1$$

$\rightarrow m_1 \ddot{q}_1 + k_1 q_1 + k_2 (q_1 - q_2) = -u_1$

$$\boxed{\ddot{q}_1 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) q_1 + \left(\frac{k_2}{m_1}\right) q_2 - \frac{U_1}{m_1}} \quad -(1)$$

$$\rightarrow \mathcal{L} = 0.5\, m_1 \dot{q}_1^2 + \underline{0.5\, m_2 \dot{q}_2^2} - 0.5\, k_1 q_1^2 - 0.5\, \underline{k_2 (q_1 - q_2)^2}$$

$q = q_2$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = \underline{Q_2}$$

$$\frac{d}{dt}\left(\cancel{0.5}\, m_2 (\cancel{2}\dot{q}_2)\right) - \left(-\cancel{0.5}\, k_2 \left[\cancel{2}(q_1 - q_2)(-1)\right]\right) = (U_1 + U_2)$$

$$m_2 \ddot{q}_2 - k_2 (q_1 - q_2) = U_1 + U_2$$

$$\boxed{\ddot{q}_2 = \frac{k_2 q_1}{m_2} - \frac{k_2 q_2}{m_2} + \frac{U_1}{m_2} + \frac{U_2}{m_2}} \quad -(2)$$

$\rightarrow x_1 = q_1 \qquad \rightarrow x_3 = q_2$

$\rightarrow x_2 = \dot{q}_1 \qquad x_4 = \dot{q}_2$

$\rightarrow \dot{x}_1 = \dot{q}_1 = x_2$

$$\dot{x}_2 = \ddot{q}_1 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) q_1 + \left(\frac{k_2}{m_1}\right) q_2 - \frac{u_1}{m_1}$$ From (1)

$$\dot{x}_2 = -\frac{(k_1+k_2)}{m_1} x_1 + \left(\frac{k_2}{m_1}\right) x_3 - \frac{u_1}{m_1}$$

$$\dot{x}_3 = \dot{q}_2 = x_4$$

$$\rightarrow \dot{x}_4 = \ddot{q}_2 = \frac{k_2 q_1}{m_2} - \frac{k_2 q_2}{m_2} + \frac{u_1}{m_2} + \frac{u_2}{m_2}$$ From (2)

$$= \frac{k_2}{m_2} x_1 - \frac{k_2 x_3}{m_2} + \frac{u_1}{m_2} + \frac{u_2}{m_2}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{1}{m_1} & 0 \\ 0 & 0 \\ \frac{1}{m_1} & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

4X1   4X4   4X1   4X2   2X1

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}}_{\substack{4\times1 \\ \dot{X}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}}_{\substack{4\times4 \\ A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times1} + \underbrace{\begin{bmatrix} 0 & 0 \\ -\frac{1}{m_1} & 0 \\ 0 & 0 \\ \frac{1}{m_1} & \frac{1}{m_2} \end{bmatrix}}_{\substack{4\times2 \\ B}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\substack{2\times1 \\ u}}$$

$$\boxed{\begin{matrix} n=4 \\ m=2 \end{matrix}}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\rightarrow \dot{X} = Ax + Bu \qquad \text{Linear equation}$$

Stability of continuous time system (uncontrolled)

$\dot{x} = Ax$

λ = eigenvalues

To check stability

① compute eigenvalues of A: $|A - \lambda I| = 0$

② If the real part of the eigenvalues are negative then the system is stable, else not

If the system is unstable, we can use control, u, to stabilize the system

Controllability

A linear system is controllable if and only if it can be transferred from any initial state $x = x_0$ to any terminal state $x = x(t)$ in finite time

Checking controllability

$\rightarrow \underline{C_0} = \underline{[A^{n-1}B \quad A^{n-2}B \quad \ldots \quad AB \quad B]}$ *

$\dot{x} = Ax + Bu \qquad A_{n\times n},\ B_{n\times m},\ X_{n\times 1},\ u_{m\times 1}$

If rank $(C_0) = n$ system is controllable ✓

rank $(C_0) < n$ system is uncontrollable ✓

pip instal control

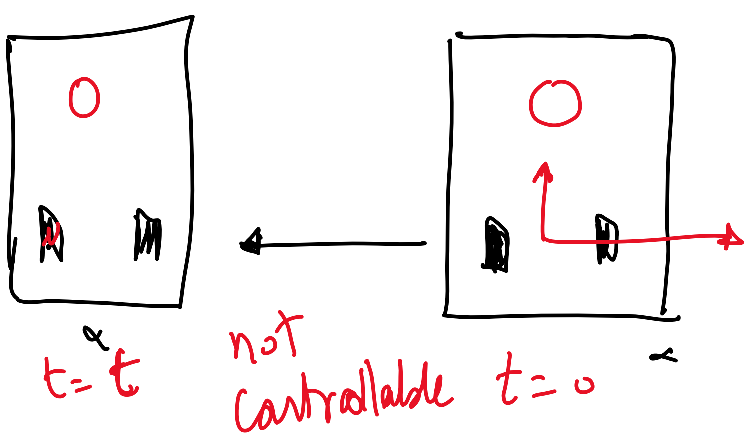
package

import control

C_0 = control.ctrb (A, B)

np. linalg. matrix_rank (C_0)

Diff-drive



$\uparrow V$

$\underline{\underline{w = 0}}$

Methods of control

① Pole Placement

We assume $u = -KX$

user chosen gain matrix

$$\dot{X} = AX + Bu$$
$$= AX + B(-KX)$$

$$\dot{X} = \underbrace{(A - BK)}X$$

$$\dot{X} = \tilde{A} X$$

Tune K such that real part of the eigenvalues of $\tilde{A} = (A - BK)$ are negative.

→ p = location of eigenvalues

$\underline{K}$ = $\underline{\text{control}}$.place (A, B, p)

user chosen

The eigenvalues of $(A - BK)$ are located at p

2) Linear Quadratic Regulator

Infinite horizon problem

$$J_{LQR} = \int_0^{\infty} \left(x^T Q x + u^T R u + 2x^T N u\right) dx$$

cost

Q, R, N are user chosen matrices

x - state

u - control

Solution (analytically obtained)

$$u = -Kx$$

$$K = -R^{-1}\left(B^T P + N^T\right) \quad \left\{\dot{x} = Ax + Bu\right\}$$

$$A^T P + PA - (PB + N) R^{-1} (B^T P + N^T) + Q = 0$$

Ricatti Eq$^{\underline{n}}$

We need to solve for P matrix such that this equation is satisfied.

P - semi-positive definite matrix

$K, P, E = \text{control.lqr}\,(A, B, \underbrace{Q, R, N}_{\text{User chosen matrices}})$

gaint

Solution to the Ricatti Equation

Eigenvalues of the closed loop system $(A-BK)$

identity

$Q_{n\times n} = I_{n\times n}$

$R_{m\times m} = \rho\, I_{m\times m}$

$N_{n\times m} = 0_{n\times n}$

improve Scaling

$$\begin{bmatrix} \frac{1}{\|x_{1,max}\|^2} & & \\ & \frac{1}{\|x_{2,max}\|^2} & \\ & & \ddots \end{bmatrix}$$

$$\dot{x} = \underset{n\times n}{A}\,\underset{n\times 1}{x} + \underset{n\times m}{B}\,\underset{m\times 1}{u}$$

$\dot{x}$: $n\times 1$

$$J = \int x^T Q x + \cancel{u^T R u} + \cancel{2x^T N u}$$

J: cost 1×1; x^T: $1\times n$; Q: $n\times n$; x: $n\times 1$; u^T: $1\times m$; R: $m\times m$; u: $m\times 1$

$$\underset{1\times n}{x^T}\ \underset{n\times m}{N}\ \underset{m\times 1}{u}$$

Linearization

LQR / Pole placement works only for linear systems : $\dot{x} = Ax + Bu$

How can we apply LQR / Pole placement to non linear systems : $\dot{x} = f(x, u)$ where

f is non-linear

Solution is to linearize the system about some operating point x_0, u_0

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x_0, u_0 — operating point

Replace $x \to x_0 + \delta x$, $u \to u_0 + \delta u$ in $\dot{x} = f(x, u)$

$$\dot{\overline{(x_0 + \delta x)}} = f(x_0 + \delta x, u_0 + \delta u)$$

Taylor series

≈ 0

$$\dot{x}_0 + \delta\dot{x} = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0, u_0} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0, u_0} \delta u + \text{higher order terms } \delta x^2, \delta u^2$$

$$\cancel{\dot{x}_0} + \delta\dot{x} = \cancel{f(x_0,u_0)} + \delta x \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} + \delta u \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}$$

Because $\dot{x}_0 = f(x_0, u_0)$

steady state

$$\delta\dot{x} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}}_{A} \delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}}_{B} \delta u$$

$$\boxed{\begin{gathered} \delta\dot{\underline{x}} = A\,\delta\underline{x} + B\,\delta u \\ A = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \end{gathered}}$$

EXAMPLE:

Consider the differential drive car dynamics

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$

State is $[x, y, \theta]$ control is $[v, \omega]$

Question: Linearize the system at some operating point $\underline{X} = [x_0, y_0, \theta_0)$ $U = [v_0, \omega_0]$

$$\delta\dot{x} = A\,\delta x + B\,\delta u$$

$A = \dfrac{\partial f}{\partial \cancel{x}}$

$B = \dfrac{\delta f}{\partial u}$ — $\begin{bmatrix} v \\ \omega \end{bmatrix}$

$$\dot{X} = f(X)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}}_{f}$$

$$f: \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial v\cos\theta}{\partial x} & \frac{\partial v\cos\theta}{\partial y} & \frac{\partial v\cos\theta}{\partial \theta} \\ \frac{\partial v\sin\theta}{\partial x} & \frac{\partial v\sin\theta}{\partial y} & \frac{\partial v\sin\theta}{\partial \theta} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial \theta} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$\underline{A} = \left.\frac{\partial f}{\partial x}\right|_{x_0, u_0} = \begin{bmatrix} 0 & 0 & -v_0\sin\theta_0 \\ 0 & 0 & v_0\cos\theta_0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{B} = \left.\frac{\partial f}{\partial u}\right|_{x_0, u_0} = \begin{bmatrix} \frac{\partial v\cos\theta}{\partial v} & \frac{\partial v\cos\theta}{\partial \omega} \\ \frac{\partial v\sin\theta}{\partial v} & \frac{\partial v\sin\theta}{\partial \omega} \\ \frac{\partial \omega}{\partial v} & \frac{\partial \omega}{\partial \omega} \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & -v_0 \sin\theta_0 \\ 0 & 0 & v_0 \cos\theta_0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \delta u$$

How to use this

$$u = \underline{u_0} + \delta u$$

$$u = \underline{\begin{bmatrix} v_0 \\ w_0 \end{bmatrix}} + \underline{\delta u} \qquad \delta u = -k\,\delta x$$

LQR / pole placement

diff-drive car

This controller is different from feedback linearization

Feedback linearization

$$\dot{x} = f(x, u) \quad \text{— car}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

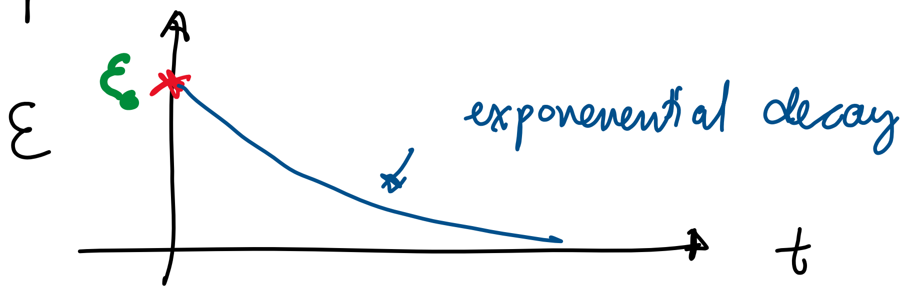
$$\dot{x} = \underbrace{\dot{x}_{ref} + k_p (x_{ref}}_{= u} - x) \quad \text{— Feedback linearization}$$

Why does this work?

$$\underbrace{\dot{x} - \dot{x}_{ref}}_{\dot{\varepsilon}} = k_p \underbrace{(x_{ref} - x)}_{-\varepsilon}$$

$$\dot{\varepsilon} = -k_p \varepsilon$$

$$\dot{\varepsilon} + k_p \varepsilon = 0 \xrightarrow{\text{sol}^n} \varepsilon = \varepsilon_0 e^{-k_p t}$$



Manipulator System

$$\underline{M(q)}\ddot{q} + \underline{C(q,\dot{q})}\dot{q} + \underline{G(q)} = \underbrace{B(q)}_{n \times m}\underset{m \times 1}{u}$$

$\dot{x} = f(x,u)$ — We have seen how to linearize this system

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$
angle

angular velocity

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left[-C(q,\dot{q})\dot{q} - G(q) + B(q)u\right] \end{bmatrix}$$

$$= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \underline{f(x,u)}$$

$$A = \frac{\partial f}{\partial x} \qquad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left[-C(q,\dot{q})\dot{q} - G(q) + B(q)u\right] \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial q & \partial f_1/\partial \dot{q} \\ \dfrac{\partial f_2}{\partial q} & \partial f_2/\partial \dot{q} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & I \\ \dfrac{\partial f_2}{\partial q}\ \text{I} & \dfrac{\partial f_2}{\partial \dot{q}}\ \text{II} \end{bmatrix}}_{\text{see below}} \quad n \times n$$

$$\frac{\partial f_2}{\partial q} = \frac{\partial}{\partial q}\left[M^{-1}\left[-C\dot{q} - G + Bu\right]\right]$$

$$= \frac{\partial M^{-1}}{\partial q}\underbrace{\left[-C\dot{q} - G + Bu\right]}_{M\ddot{q} \approx 0 \text{ steady state}} + M^{-1}\underbrace{\left[\frac{\partial C}{\partial q}\dot{q} - \frac{\partial G}{\partial q} + \frac{\partial B u}{\partial q}\right]}_{\neq 0 \quad \neq 0 \quad \neq 0}$$

$$\frac{\partial f_2}{\partial q} = M^{-1}\left[\frac{\partial C}{\partial q}\dot{q} - \frac{\partial G}{\partial q} + \frac{\partial B}{\partial q}u\right] \quad - \text{I}$$

$$\frac{\partial f_2}{\partial \dot q} = \frac{\partial}{\partial \dot q}\left[\underline{M^{-1}(q)}\left[-C(q,\dot q)\dot q - G(q) + B(q)u\right]\right]$$

$$= \cancel{\frac{\partial M^{-1}(q)}{\partial \dot q}}^{\,0} \left(-C(q,\dot q)\dot q - G(q) + B(q)u\right)$$

$$+ M^{-1}(q)\left[-\frac{\partial C(q,\dot q)\dot q}{\partial \dot q} - \cancel{\frac{\partial G(q)}{\partial \dot q}}^{\,0} + \cancel{\frac{\partial B(q)}{\partial \dot q}}^{\,0} u\right]$$

$$\frac{\partial f_2}{\partial \dot q} = M^{-1}(q)\left(-\frac{\partial C(q,\dot q)\dot q}{\partial \dot q}\right) \quad - \text{II}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix} = \begin{bmatrix} \cancel{\frac{\partial}{\partial u}\dot q}^{\,0} \\ \frac{\partial}{\partial u}\left[\underline{M^{-1}}\left(-\cancel{C\dot q} - \cancel{G} + \underline{B(q)}u\right)\right] \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}(q)\,B(q) \end{bmatrix}$$

$$\delta\dot x = A\,\delta x + B\,\delta u$$

$$A = \begin{bmatrix} 0 & I \\ +M^{-1}\left(-\underline{\frac{\partial C\dot q}{\partial q}} - \frac{\partial G}{\partial q} + \frac{\partial B}{\partial q}u\right) & M^{-1}\left(-\underline{\frac{\partial C\dot q}{\partial \dot q}}\right) \end{bmatrix}$$

identity matrix

$\dot q \simeq 0$ is zero if $\dot q = 0$ //

$\dot q \simeq 0$ is zero if $\dot q = 0$ //

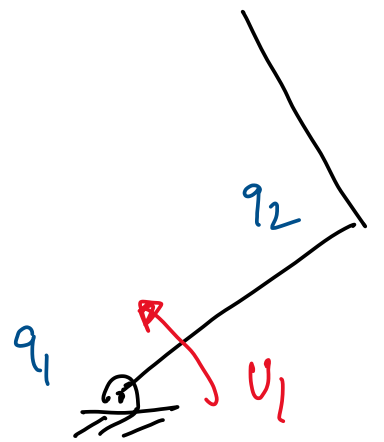
$\dot{q}=0$ //

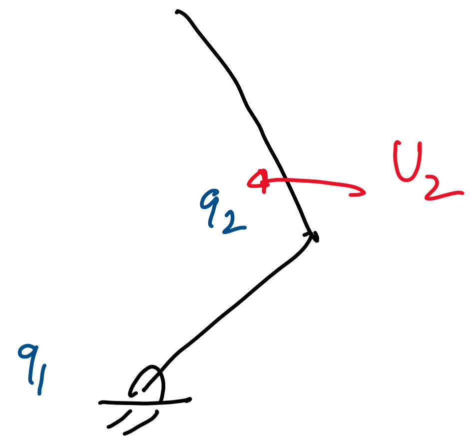
if $\dot{q}=0$ //

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ M^{-1} B(q) \end{bmatrix}$$

EXAMPLE: Underactuated Double Pendulum

①





Only 1 actuator at q_1	Only 1 actuator at q_2
2 degrees of freedom (DOF)	2 degrees of freedom
$1 < 2$	$1 < 2$
# actuators < DOF	# actuators < DOF
underactuated	underactuated
Pendubot	**Acrobot**

① Linearize

② <u>LQR</u> $u = -k_1 q_1 - k_2 q_2 - k_3 \dot{q}_1 - k_4 \dot{q}_2$