Feedback Control of Manipulators

Equations of motion

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j}$$

$A\ddot{\theta} = b$

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau}$$

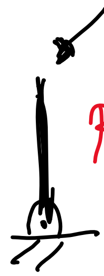
$M(q)$ — mass matrix
$C(q,\dot{q})\dot{q}$ — coriolis acceleration / torque
$G(q)$ — gravity torque
τ — external torque

$$\underbrace{M(q)\ddot{q}} = \underbrace{\tau - C(q,\dot{q})\dot{q} - G(q)}$$

$$\boxed{A\ddot{\theta} = b}$$

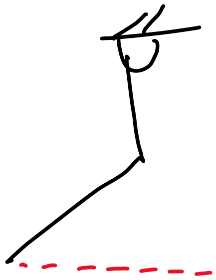
Two objective for control

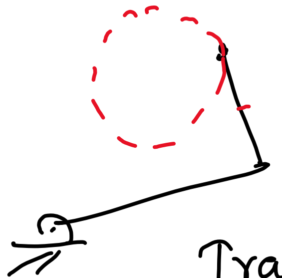
① Set-point control: e.g.



② Trajectory tracking control

Robot Kicking a ball

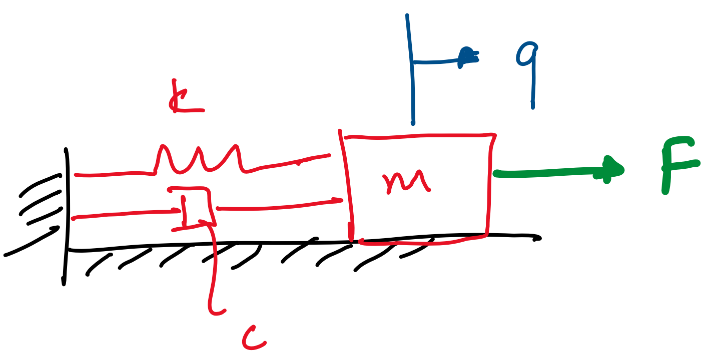




Track a circle

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$$m\ddot{q} + c\dot{q} + kq = F$$ ~ simple example



Spring mass damper system

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = \cancel{\frac{F}{m}} \; 0$$

Free Vibrations

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \qquad 2\xi\omega_n = \frac{c}{m}$$

$$\boxed{\xi = \frac{c}{2\sqrt{mk}}}$$

Xi

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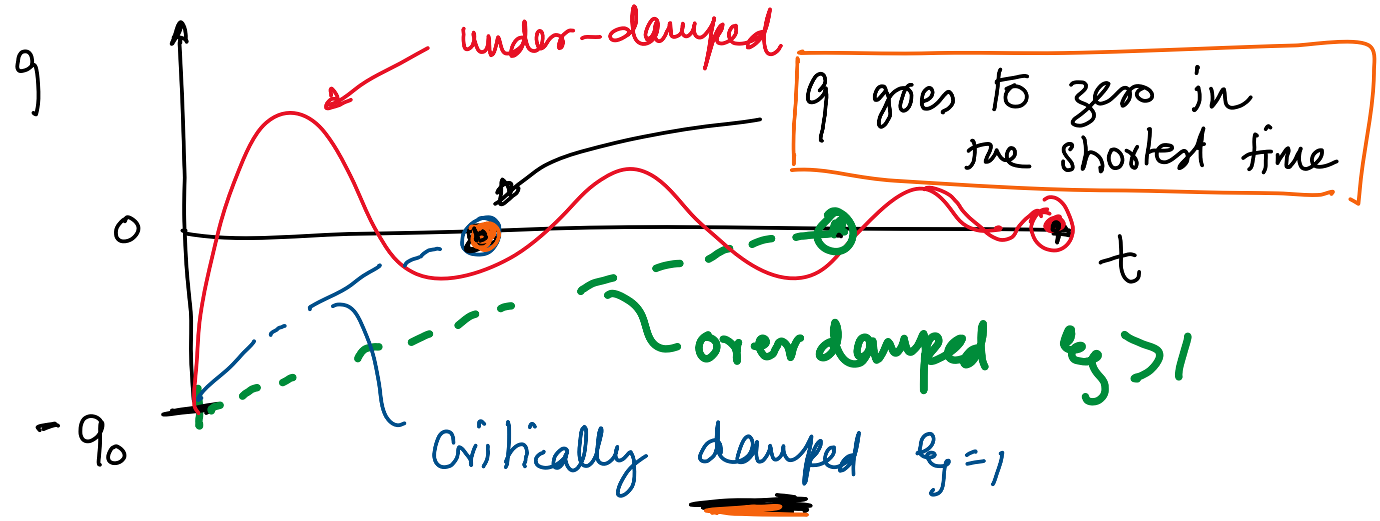
3 cases

① $\xi > 1 \Rightarrow c > 2\sqrt{mk}$ Overdamped

② $\xi = 1 \Rightarrow c = 2\sqrt{mk}$ critically damped

③ $\xi < 1 \Rightarrow c < 2\sqrt{mk}$ under damped



$$m\ddot{q} + c\dot{q} + kq = F$$

We will design a feedback controller $F(q,\dot{q})$ such that the system is critically damped

$$F = -k_p q - k_d \dot{q}$$

proportional-derivative controller

$(-k_p q)$ $(-k_d \dot{q})$

k_p, k_d — are user-chosen gains.

$$m\ddot{q} + c\dot{q} + kq = \overbrace{-k_p q - k_d \dot{q}}^{F}$$

$$\rightarrow m\ddot{q} + (c + k_d)\dot{q} + (k + k_p) q = 0$$

Critically damped

$$(c + k_d) = 2\sqrt{m(k + k_p)} \quad \left(\sim c = 2\sqrt{mk}\right)$$

Solve for k_d ; keep k_p = fixed

Squaring both sides

$$(c + k_d)^2 = 4m(k + k_p)$$

$$c^2 + 2ck_d + k_d^2 = 4mk + 4mk_p$$

$$\underline{1}k_d^2 + \underline{2c}k_d + \underline{(c^2 - 4mk - 4mk_p)} = 0$$

Solve for k_d

$$k_d = \frac{-2c \pm \sqrt{(2c)^2 - 4(1)(c^2 - 4mk - 4mk_p)}}{2(1)}$$

Take +ive root

$$\boxed{k_d = -c + 2\sqrt{(k+k_p)m}}$$

$k_p, k_d \rightarrow$ designer's choice

Extend to a 2D system

1D $m\ddot{q} + c\dot{q} + kq = F$ ✓

2D

$$\left\{ \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\}$$

$$\rightarrow \underline{F} = -\underline{K_p}\, q - \underline{K_d}\, \dot{q}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = -\underbrace{\begin{bmatrix} K_{p11} & K_{p12} \\ K_{p21} & K_{p22} \end{bmatrix}}_{\text{4 parameter}} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \underbrace{\begin{bmatrix} K_{d11} & K_{d12} \\ K_{d21} & K_{d22} \end{bmatrix}}_{\text{4 parameters}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

8 parameters

2 critically damped conditions.

It is difficult to tune 8 parameters with only 2 conditions.

$\Rightarrow$ Feedback Linearization

Feedback Linearization / Control partitioning

$$\underline{M}\ddot{q} + \underbrace{C(q,\dot{q})}\dot{q} + \underline{G}(q) = \underline{\tau} \quad - \textcircled{1}$$

$$\boxed{\underline{\tau} = \underline{\hat{M}}(-K_p q - K_d \dot{q}) + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) \quad - \textcircled{2}}$$

$\hat{M}, \hat{C}, \hat{G} \rightarrow$ estimates of M, C, G

Lets assume $M = \hat{M}$, $C = \hat{C}$, $G = \hat{G}$

Substitute ② in ①

$$M\ddot{q} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{G(q)} = M(-K_p q - K_d \dot{q}) + \cancel{C(q,\dot{q})q} + \cancel{G(q)}$$

$$M(\ddot{q} + K_d \dot{q} + K_p q) = 0$$

$M \neq 0 \qquad \boxed{\ddot{q} + K_d \dot{q} + K_p q = 0} \leftarrow$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} K_{d_1} & 0 & 0 & \cdots \\ 0 & K_{d_2} & 0 & \cdots \\ & \ddots & & \\ 0 & 0 & \cdots & K_{d_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} K_{p_1} & 0 & \cdots \\ 0 & K_{p_2} & \cdots \\ \vdots & & \\ 0 & \cdots & K_{p_n} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = 0$$

These equations are decoupled

$$\left.\begin{aligned}
&\ddot{q}_1 + \underline{k_{d_1}}\,\dot{q}_1 + \underline{k_{p_1}}\,q_1 = 0\\
&\ddot{q}_2 + k_{d_2}\,\dot{q}_2 + k_{p_2}\,q_2 = 0\\
&\quad\vdots\\
&\ddot{q}_n + k_{d_n}\,\dot{q}_n + k_{p_n}\,q_n = 0
\end{aligned}\right\}$$

n decoupled equations

NOTE: $\underline{m}\ddot{q} + (k_d + \underline{c})\dot{q} + (\underline{k_p} + \underline{k})q = 0$

Critically damped: $\underline{k_d} = -c + 2\sqrt{(k + k_p)\,\underline{m}}$

Derived earlier

$$k_{d_1} = 0 + 2\sqrt{(0 + k_{p_1})\,1}$$

$$k_{d_1} = 2\sqrt{k_{p_1}}$$

$$\vdots$$

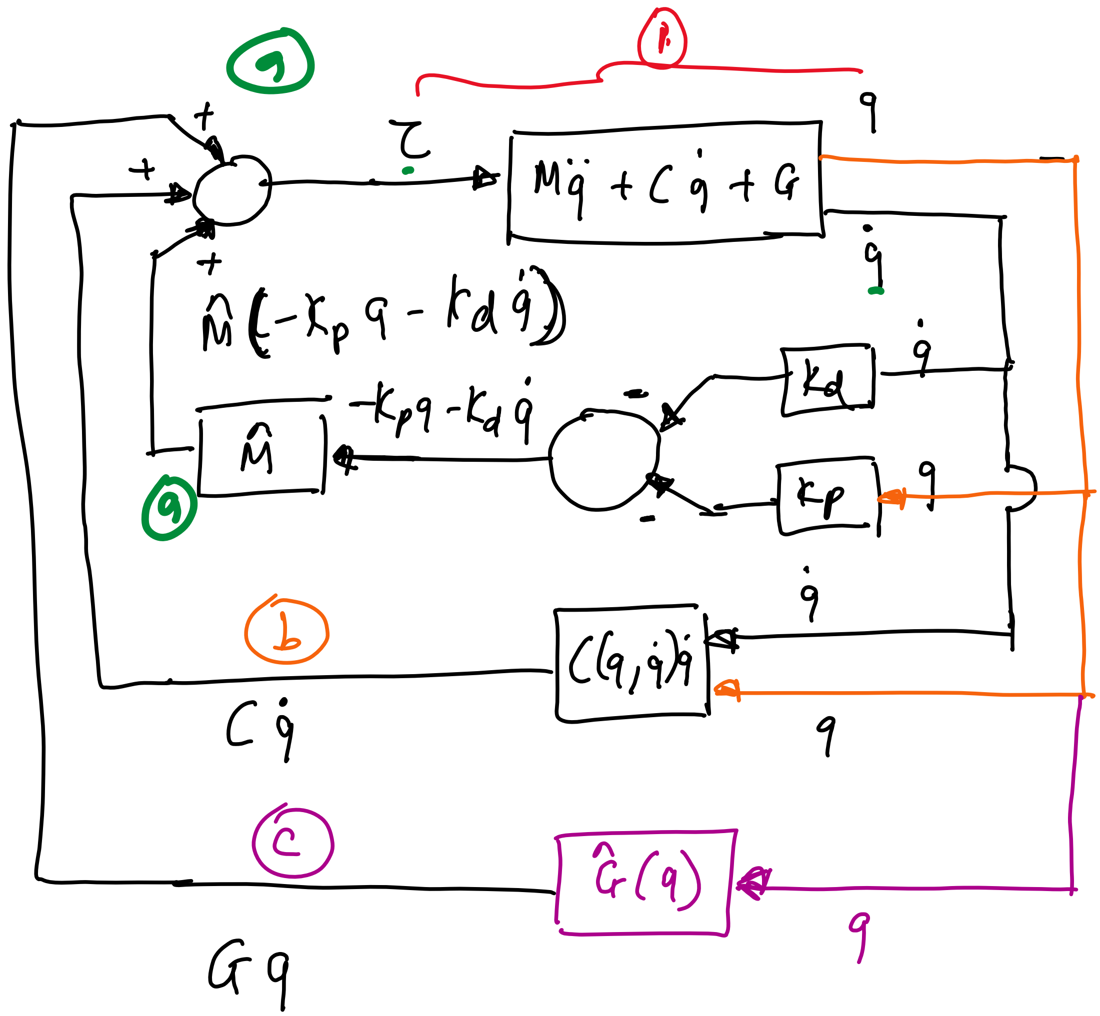
$$k_{d_n} = 2\sqrt{k_{p_n}}$$

$$\boxed{k_{d_i} = 2\sqrt{k_{p_i}}}$$

Block Diagram

$$\checkmark\ M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \qquad -①$$

$$\checkmark\ \tau = \underbrace{\hat{M}(-k_p q - k_d \dot{q})}_{ⓐ} + \underbrace{\hat{C}(q,\dot{q})\dot{q}}_{ⓑ} + \underbrace{\hat{G}(q)}_{ⓒ} \qquad -②$$



$q, \dot{q} \longrightarrow$ outputs (sensor)

$\tau \longrightarrow$ input (actuator)

Code Example !

control_partitioning_pd

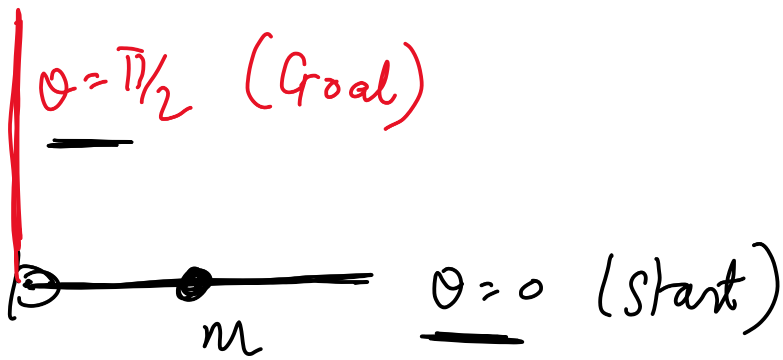
$$\underset{2\times 2}{M}\,\underset{2\times 1}{\ddot{q}} + \underset{2\times 2}{C}\,\underset{2\times 1}{\dot{q}} + \underset{2\times 2}{G}\,\underset{2\times 1}{q} = \underset{2\times 1}{F}$$

① $\cancel{F = -k_p q - k_d \dot{q}}$ ✗

② $F = \underline{M}\left(-k_p(q - q_{des}) - k_d \dot{q}\right) + \underline{C}\dot{q} + \underline{G}q$ ✓

③ $F = \underline{\hat{M}}\left(-k_p(q - q_{des}) - k_d \dot{q}\right) + \underline{\hat{C}}\dot{q} + \underline{\hat{G}}q$

Example: 1-link pendulum



$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$$(I + ml^2)\ddot{\theta} + 0 + \underbrace{mgl}\sin\theta = \tau$$

Controllers

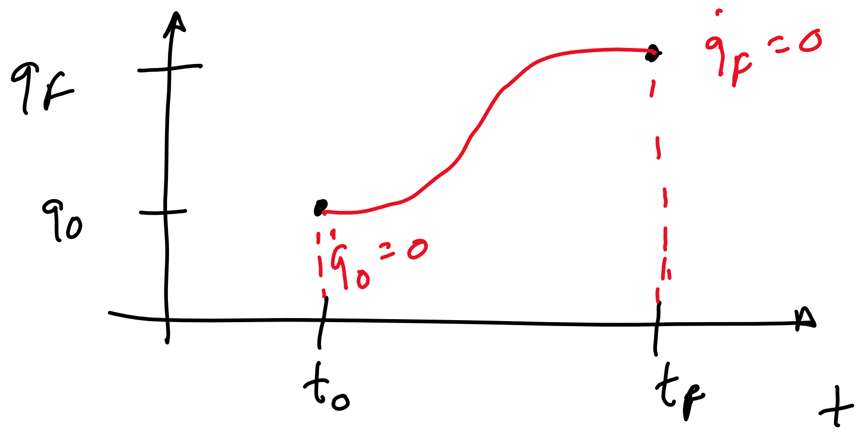
✓ ① $\tau = -K_p(q - \overset{(\pi/2)}{q_{des}}) - K_d\dot{q}$ ✓

✓ ② $\tau = \underset{(I+ml^2)}{M}(-K_p(q - \underset{(\pi/2)}{q_{des}}) - K_d\dot{q}) + \underset{0}{C(q,\dot{q})\dot{q}} + \underset{mgl\sin\theta}{G(q)}$ ✓

(3a_ pendulum _pd) – This folder

(3b_pendulum_control _partitioning)

Trajectory tracking in joint space



✓ $q_{ref}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ ✓

Dynamics $\quad M\ddot{q} + C\dot{q} + G = \tau \quad$ — (1)

Control: $\quad \tau = M\left(\ddot{q}_{ref} - k_d(\dot{q} - \dot{q}_{ref}) - k_p(q - q_{ref})\right) + \ldots$

$$+ \; G(q) + C(q,\dot{q})\dot{q} \quad \text{— (2)}$$

Why does this τ work?

Substitute (2) in (1)

$$M\ddot{q} + \cancel{C\dot{q}} + \cancel{G} = M(\ddot{q}_{ref} - k_d(\dot{q} - \dot{q}_{ref}) - k_p(q - q_{ref})$$
$$+ \cancel{C\dot{q}} + \cancel{G}$$

$$M\ddot{q} + \cancel{C\dot{q}} + \cancel{G} = M(\ddot{q}_{ref} - K_d(\dot{q} - \dot{q}_{ref}) - K_p(q - q_{ref})) + \cancel{C\dot{q}} + \cancel{G}$$

$$M\ddot{q} = M(\ddot{q}_{ref} - K_d(\dot{q} - \dot{q}_{ref}) - K_p(q - q_{ref}))$$

$$M\left(\underbrace{(\ddot{q} - \ddot{q}_{ref}) + K_d(\dot{q} - \dot{q}_{ref}) + K_p(q - q_{ref})}\right) = 0$$

$$M \neq 0$$

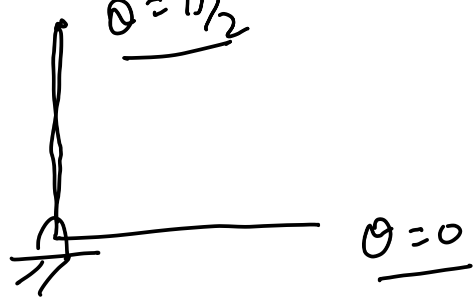
$$(\ddot{q} - \ddot{q}_{ref}) + K_d(\dot{q} - \dot{q}_{ref}) + K_p(q - q_{ref}) = 0$$

$$e = q - q_{ref}$$

$$\rightarrow \underset{n\times1}{\underline{\ddot{e}}} + K_d\,\underset{n\times1}{\underline{\dot{e}}} + K_p\,\underset{n\times1}{\underline{e}} = 0 \longleftarrow$$

similar to spring-mass damper equ.

EXAMPLE: Trajectory Tracking of a 1-link pendulum



$$\begin{matrix} \theta = 0 \\ t = 0 \end{matrix} \longrightarrow \begin{matrix} \theta = \pi/2 \\ t = t_1 \end{matrix} \qquad \theta_{1,ref} = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$\begin{matrix} \theta = \pi/2 \\ t = t_1 \end{matrix} \longrightarrow \begin{matrix} \theta = 0 \\ t = t_2 \end{matrix} \qquad \theta_{2,ref} = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$
$$ml^2 \ddot{\theta} + 0 + mg \sin\theta = T$$

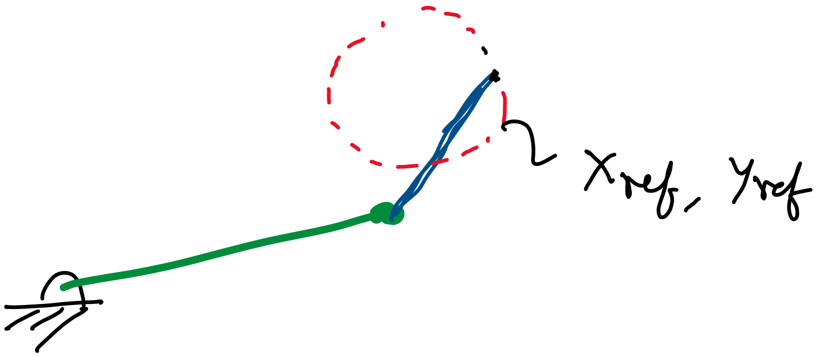
Controller

$$\tau = \underbrace{M}_{ml^2}\left(\ddot{\theta}_{ref} - K_p(\theta - \theta_{ref}) - K_d(\dot{\theta} - \dot{\theta}_{ref})\right) + \underbrace{C(q,\dot{q})\dot{q}}_{0} + \underbrace{G(q)}_{mgl \sin\theta}$$

<4a_pendulum_trajectory_tracking>
<4b_double pendulum_trajectory_tracking> } PYTHON

Feedback linearization in the Task space

eg



Here $x_{ref}(t)$, $y_{ref}(t)$ is given

How to control the joints $\tau = ?$

Feedback linearization in joint space

$$\tau = M\left(\ddot{q}_{ref} - K_p(q - q_{ref}) - K_d(\dot{q} - \dot{q}_{ref})\right) + C(q,\dot{q})\dot{q} + G(q)$$

joint space or θ

$$X = \begin{bmatrix} x_{ref} \\ y_{ref} \end{bmatrix} = f(q)$$ Forward kinematics

$$q = f^{-1}(X)$$ Inverse kinematics

$$\boxed{q_{ref} = f^{-1}(X_{ref})}$$ — I

f^{-1} obtained using fsolve

$$X = f(q)$$

$$\dot{X} = \underbrace{\frac{\partial f}{\partial q}} \dot{q} \qquad \text{(Chain rule)}$$

$$\dot{x} = J \dot{q}$$

$$\dot{X}_{ref} = J \dot{q}_{ref}$$

$$\boxed{\dot{q}_{ref} = J^{-1} \dot{x}_{ref} = J^{-1} \begin{bmatrix} \dot{x}_{ref} \\ \dot{y}_{ref} \end{bmatrix}} \quad \text{— II}$$

$$\ddot{X} = \dot{J} \dot{q} + J \ddot{q}$$

$$\ddot{X}_{ref} = \dot{J} \dot{q}_{ref} + J \ddot{q}_{ref}$$

$$\boxed{\ddot{q}_{ref} = J^{-1} \left[\ddot{X}_{ref} - \dot{J} \dot{q}_{ref} \right]} \quad \text{— III}$$

From I, II, III we have computed $q_{ref}, \dot{q}_{ref}, \ddot{q}_{ref}$ from $X_{ref}, \dot{X}_{ref}, \ddot{X}_{ref}$

$$\tau = M(\ddot{q}_{ref} - K_p(q - q_{ref}) - K_d(\dot{q} - \dot{q}_{ref}))$$

<5-double pendulum_cartesian_control> PYTHON