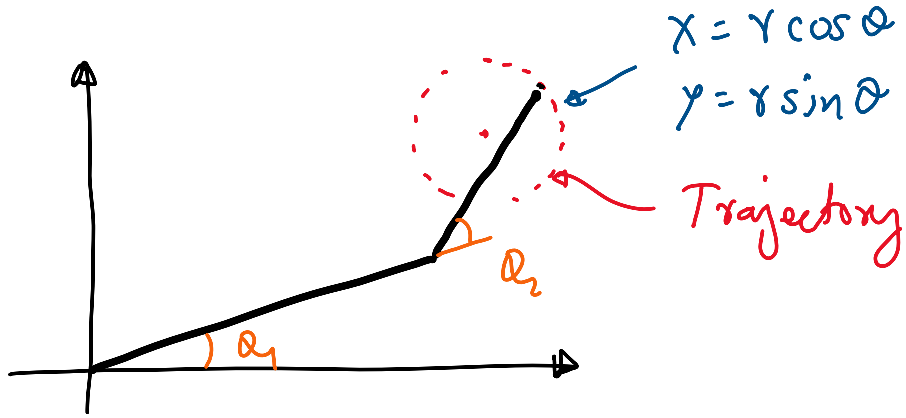
Trajectory generation



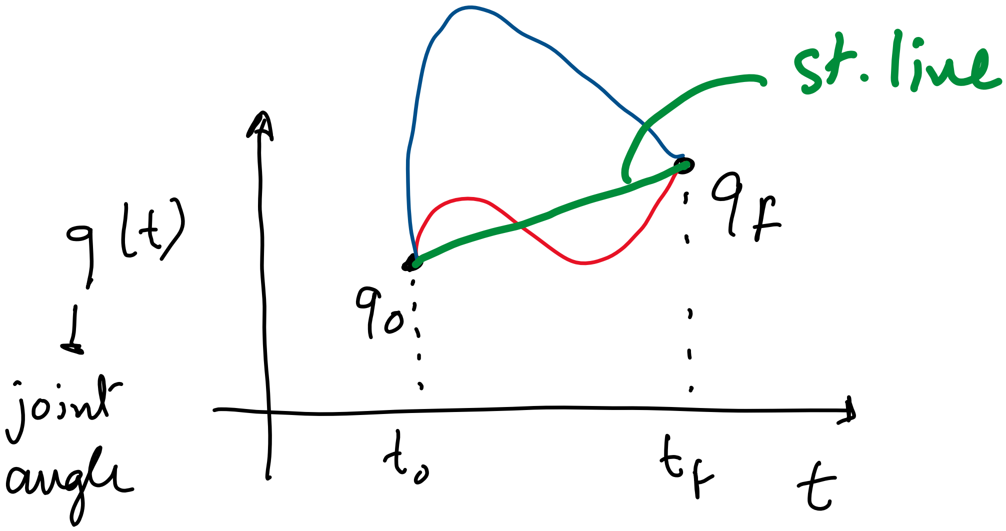
$\underline{\theta_1}$, $\underline{\theta_2}$ $\Rightarrow$ Inverse kinematics

Static problem

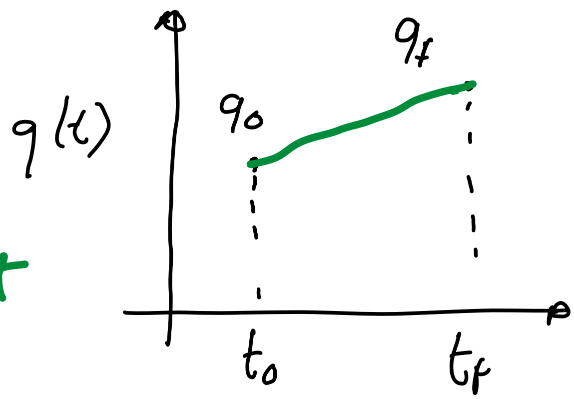
This example looked at spatial trajectory x, y
But it ignored the temporal trajectory

$x(t), y(t)$

We will look at dynamic trajectories
$x(t)$, $y(t)$, $\theta_1(t)$, $\theta_2(t)$ . . .



Connect q_0 to q_f with a straight line $q(t)$



$$q(t) = \underbrace{a_0 + a_1 t}_{\text{eqn of a straight line}}$$

a_0, a_1 are unknown constants

$$\left.\begin{array}{llll} q = q_0 & \text{at} & t = t_0 & \Rightarrow q_0 = a_0 + a_1 t_0 \\ q = q_f & \text{at} & t = t_f & \Rightarrow q_f = a_0 + a_1 t_f \end{array}\right\}$$

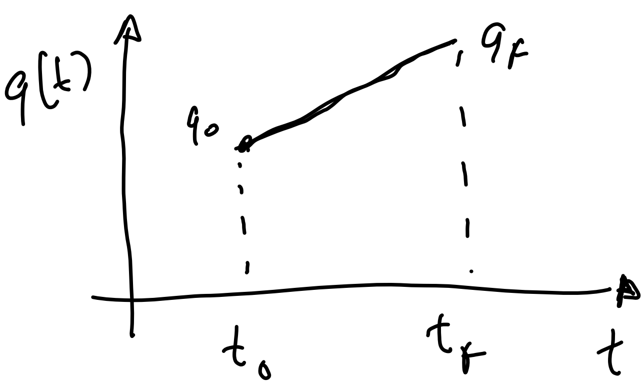
$$\underbrace{\begin{bmatrix} 1 & t_0 \\ 1 & t_f \end{bmatrix}}_{A \checkmark} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{a\,?} = \underbrace{\begin{bmatrix} q_0 \\ q_f \end{bmatrix}}_{b \checkmark}$$

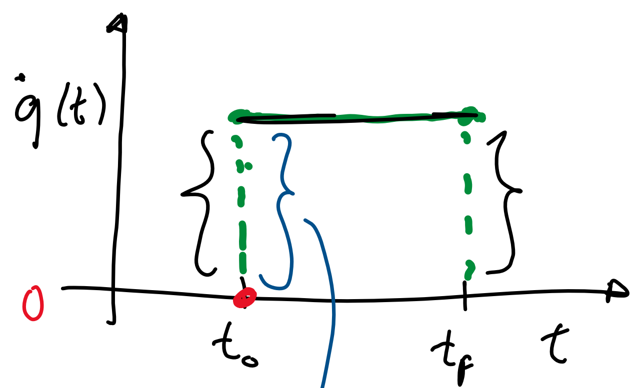
$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & t_0 \\ 1 & t_f \end{bmatrix}^{-1} \begin{bmatrix} q_0 \\ q_f \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \dfrac{q_0 t_f - q_f t_0}{t_f - t_0} \\[2ex] \dfrac{q_f - q_0}{t_f - t_0} \end{bmatrix}$$

$$q(t) = \underbrace{\left(\frac{q_0 t_f - q_f t_0}{t_f - t_0}\right)}_{a_0} + \underbrace{\left(\frac{q_f - q_0}{t_f - t_0}\right)}_{a_1} t$$

$$\dot{q}(t) = \left(\frac{q_f - q_0}{t_f - t_0}\right)$$ <u>constant</u>





jump in the velocity would damage the motor

$$\begin{cases} q(t_0) = q_0 \\ q(t_f) = q_f \end{cases}$$

$$\begin{cases} \dot{q}(t_0) = 0 \\ \dot{q}(t_f) = 0 \end{cases}$$

} ensures the velocity does not change abruptly

4 conditions

$$q(t) = \underbrace{\underline{a_0} + \underline{a_1}\, t + \underline{a_2}\, t^2 + \underline{a_3}}_{4 \text{ constants}}\, t^3 \checkmark$$

$$\rightarrow \dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\begin{cases} q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\ q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ \dot{q}(t_0) = 0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\ \dot{q}(t_f) = 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

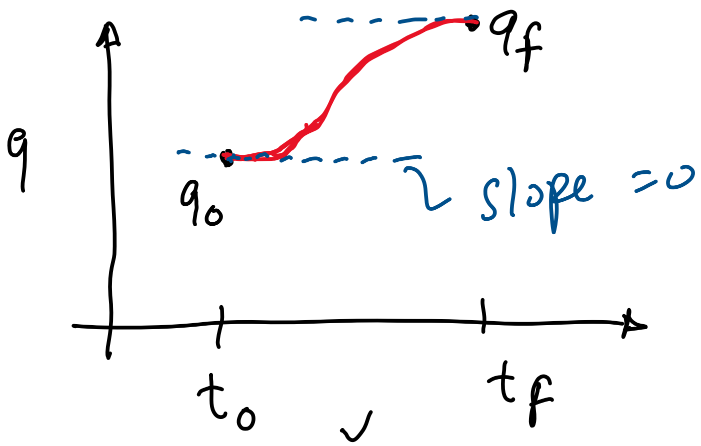
4 equations
4 unknowns

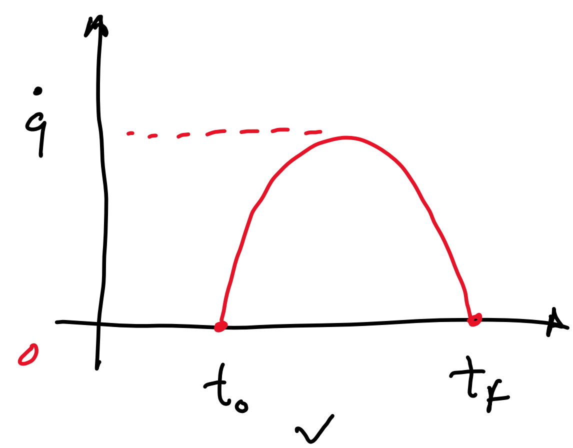
$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_f \\ 0 \\ 0 \end{bmatrix}$$

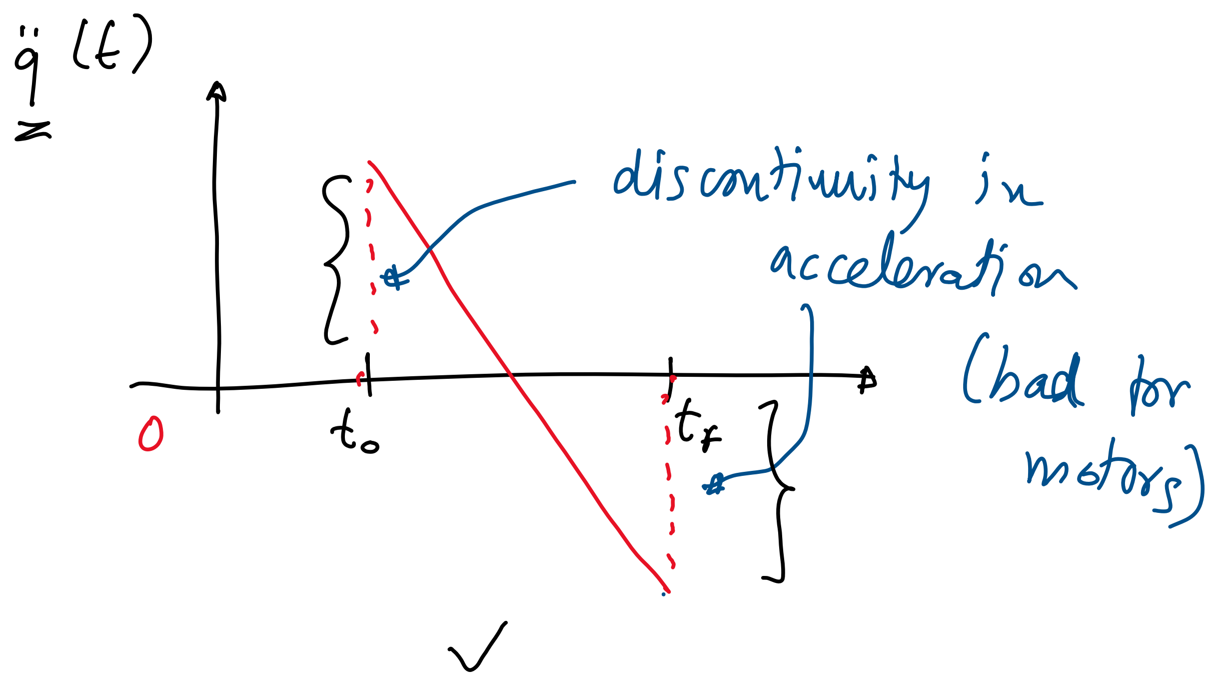
$$\begin{bmatrix} 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_3 \end{bmatrix} \; ? \; \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}^{-1} \begin{bmatrix} q_0 \\ q_f \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} q(t) \ \checkmark \\ \dot{q}(t) \ \checkmark \end{array}\right\} \text{sympy code}$$







To avoid this add 2 more conditions

$$\left.\begin{aligned} q(t_0) &= q_0 \\ q(t_f) &= q_f \\ \dot{q}(t_0) &= 0 \\ \dot{q}(t_f) &= 0 \\ \ddot{q}(t_0) &= 0 \\ \ddot{q}(t_f) &= 0 \end{aligned}\right\} \text{6 conditions}$$

$$q(t) = \underbrace{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5}_{\text{6 constants}} t^5$$

$\dddot{q}(t)$ (jerk) is discontinuous at
$t = t_0$ & $t = t_f$

Solution $\dddot{q}(t_0) = \dddot{q}(t_f) = 0$

7^{th} order polynomial

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7$$

$\frac{d^4 q}{dt^4}$ (snap)

$\frac{d^5 q}{dt^5}$ (crackle)

$\frac{d^6 q}{dt^6}$ (pop)

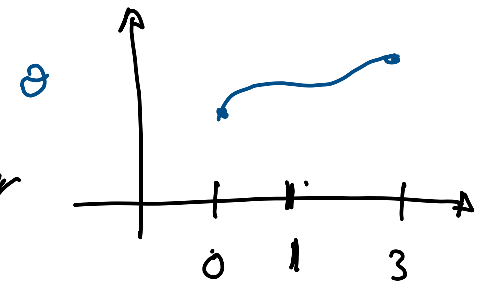
θ

Example 1: Find a time based parameterization for a revolute joint of a manipulator. The joint should move from 0 to 0.5 rad from time t= 0 to t=1 sec followed by movement from 0.5 rad to 1 rad in from t=1 to t=3 secs. Also, the velocity of the joint at the start of motion (t=0) and end of motion (t=3) should be 0 and the velocity of the joint at the intermediate point (t=1) should be 0.2 rad/s. Assume two minimal order polynomials of time, one for each movement.

$\theta(t=0) = 0$ $\qquad$ $\dot{\theta}(t=0) = 0$ ✓

$\theta(t=1) = 0.5$ $\qquad$ $\dot{\theta}(t=3) = 0$

$\theta(t=3) = 1$ $\qquad$ $\dot{\theta}(t=1) = 0.2$



6 conditions

$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$ ✓ (NOT asked)

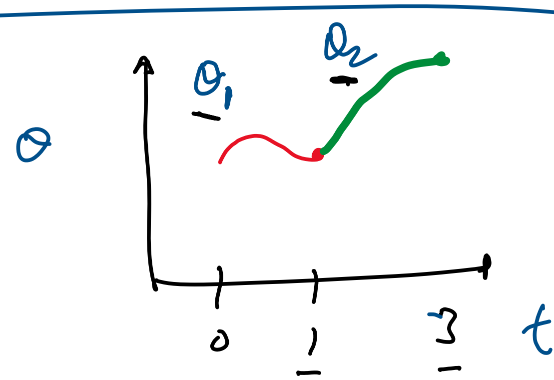
$\theta_1(t=0) = 0$

$\theta_1(t=1) = 0.5$

$\dot{\theta}_1(t=0) = 0$

$\dot{\theta}_1(t=1) = 0.2$

θ_1 $\quad 0 \le t \le 1$



4 conditions || 3rd order polynomial

$\theta_2(t=1) = 0.5$

$\theta_2(t=3) = 1$

$\dot{\theta}_2(t=3) = 0$

$\dot{\theta}_2(t=1) = 0.2$

θ_2 $\quad 1 \le t \le 3$

4 conditions || 3rd order polynomial

$$\theta_1(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3 \qquad 0 \le t \le 1$$

$$\theta_2(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3 \qquad 1 \le t \le 3$$

$$\dot{\theta}_1(t) = a_{11} + 2a_{12}t + 3a_{13}t^2 \qquad 0 \le t \le 1$$

$$\dot{\theta}_2(t) = a_{21} + 2a_{22}t + 3a_{23}t^2 \qquad 1 \le t \le 3$$

$$\theta_1(t=0) = a_{10} = 0$$

$$\theta_1(t=1) = a_{10} + a_{11} + a_{12} + a_{13} = 0.5$$

$$\dot{\theta}_1(t=0) = a_{11} = 0$$

$$\dot{\theta}_1(t=1) = a_{11} + 2a_{12} + 3a_{13} = 0.2$$

$$\theta_2(t=1) = a_{20} + a_{21} + a_{22} + a_{23} = 0.5$$

$$\theta_2(t=3) = a_{20} + 3a_{21} + 9a_{22} + 27a_{23} = 1$$

$$\dot{\theta}_2(t=3) = a_{21} + 6a_{22} + 27a_{23} = 0$$

$$\dot{\theta}_2(t=1) = a_{21} + 2a_{22} + 3a_{23} = 0.2$$

8 equations

8 unknowns

$$
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}}_{a}
=
\underbrace{\begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0.2 \\ 0.5 \\ 1 \\ 0 \\ 0.2 \end{bmatrix}}_{b}
$$

$$A a = b$$

$$a = A^{-1} b$$

Using python

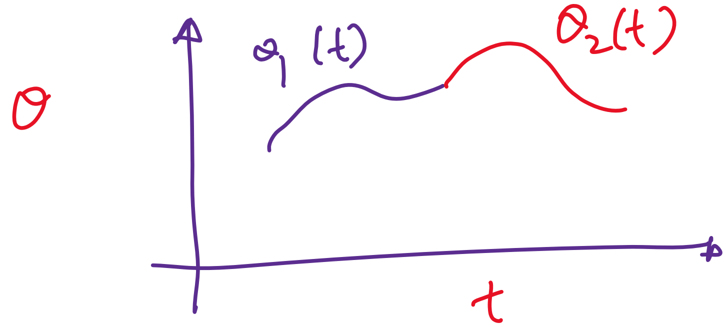
$$
a = \begin{array}{cccccccc} [0 & 0 & 1.3 & -0.8 & 0.55 & -0.375 & 0.4 & -0.075] \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{20} & a_{21} & a_{22} & a_{23} \end{array}
$$

$$\theta_1(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$\theta_2(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

θ

Example 2: Find a time based parameterization for a revolute joint of a manipulator. The joint should move from 0 to 0.5 rad from time t= 0 to t=1 sec followed by movement from 0.5 rad to 1 rad in from t=1 to t=3 secs. Also, the velocity of the joint at the start of motion (t=0) and end of motion (t=3) should be 0 and the acceleration of the joint at the intermediate point (t=1) should be continuous. Assume two minimal order polynomials of time, one for each movement.



$$3 \left\{ \begin{array}{l} \theta_1(t=0) = 0 \\ \theta_1(t=1) = 0.5 \\ \dot{\theta}_1(t=0) = 0 \end{array} \right.$$

$$\left. \begin{array}{l} \theta_2(t=1) = 0.5 \\ \theta_2(t=3) = 1 \\ \dot{\theta}_2(t=3) = 0 \end{array} \right\} 3$$

$$2 \left\{ \begin{array}{ll} \ddot{\theta}_1(t=1) = \ddot{\theta}_2(t=1) & \text{continuous acceleration} \\ \dot{\theta}_1(t=1) = \dot{\theta}_1(t=1) & \text{continuous velocity} \end{array} \right.$$

8 conditions : 4 for θ_1 and 4 for θ_2

$$\theta_1(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$\theta_2(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

Set up $A\underline{a} = b$ (do this at home)