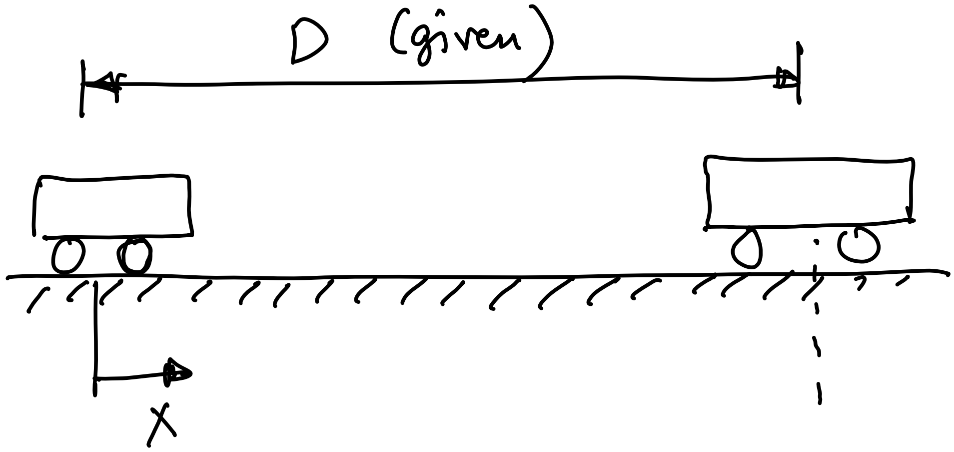
Trajectory optimization



$x(t=0)=0$ $\qquad\qquad$ $x(t=T)=D$

$\dot{x}(t=0)=0$ $\qquad\qquad$ $\dot{x}(t=T)=0$

Model: $\ddot{x} = u$ $\qquad$ Solve $u(t)=?$

$$-5 \le u \le 5$$

Goal: Minimize time (T)

Formulation

$$\min_{T,u} \int_0^T dt = T$$ cost

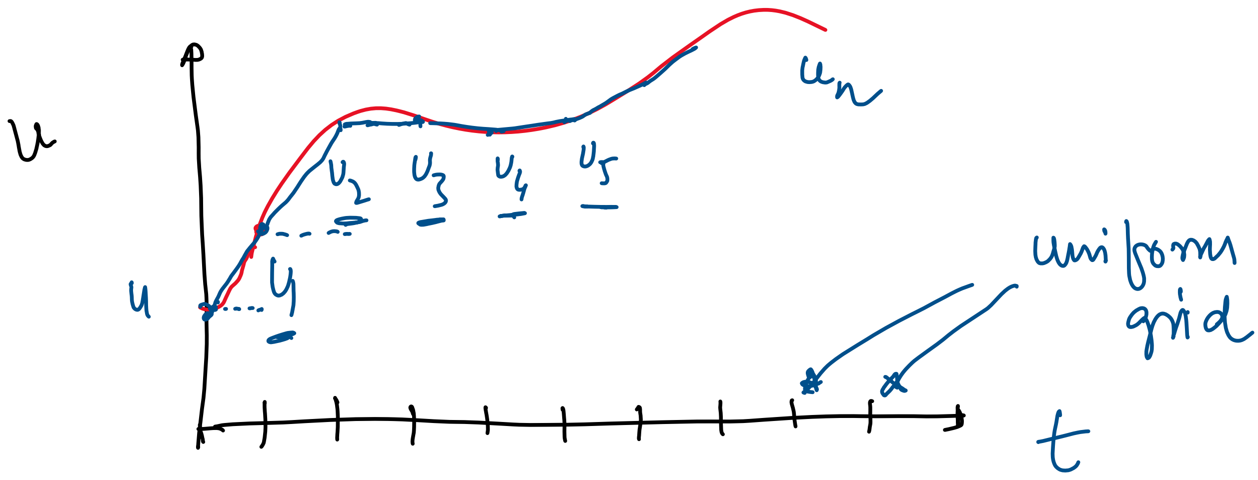
Constraint $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$ $\ddot{x} = u$

x_1 = position

x_2 = velocity

Bounds $-5 \leq u \leq 5$

Constraints
$$\begin{cases} x_1(t=0) = 0 & \text{init position} \\ x_2(t=0) = 0 & \text{init velocity} \\ x_1(t=T) = D & \text{final position} \\ x_2(t=T) = 0 & \text{final velocity} \end{cases}$$



Convert the infinite dimensional problem into finite dimension using discretization

$\underbrace{\text{discretization}}$ uniform grid

Solve for a given N (= grid points)

True solution, make N big enough

Two methods

① Collocation method

② Shooting method.

① Collocation Method

Satisfy the dynamics at grid points

$$\ddot{x} = u \rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{array}$$

a) Optimization variables

T, $u(i)$ $0 \le i \le N$ $N+1$

1

$x_1(i)$ $0 \le i \le N$ $N+1$

$x_2(i)$ $0 \le i \le N$ $N+1$

Total optimization variable $1 + 3(N+1)$

$3N + 4$

b) Cost

minimize T ✓

c) Constraints

$$\dot{x}_1 = x_2 \Rightarrow \frac{x_1(t+\Delta t) - x_1(t)}{\Delta t} = x_2(t)$$ Euler's method

$$x_1(t+\Delta t) = x_1(t) + \Delta t\, x_2(t)$$

Similarly $\dot{x}_2 = u$

$$x_2(t+\Delta t) = x_2(t) + \Delta t\, u(t)$$

$$x_1(i+1) = x_1(i) + \Delta t\, x_2(i) \quad \Big\| \quad \underbrace{0 \le i \le N-1}_{N}$$

$$x_2(i+1) = x_2(i) + \Delta t\, u(i) \quad \Big\| \quad \underbrace{0 \le i \le N-1}_{N}$$

$$\left.\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_1(N) &= D \\ x_2(N) &= 0 \end{aligned}\right\} 4 \rightarrow (N+N+4) = (2N+4)$$

$-5 \le u(i) \le 5 \rightsquigarrow$ Put them as bounds for the optimization

We 3N+4 optimization variables

But ONLY 2N+4 constraints

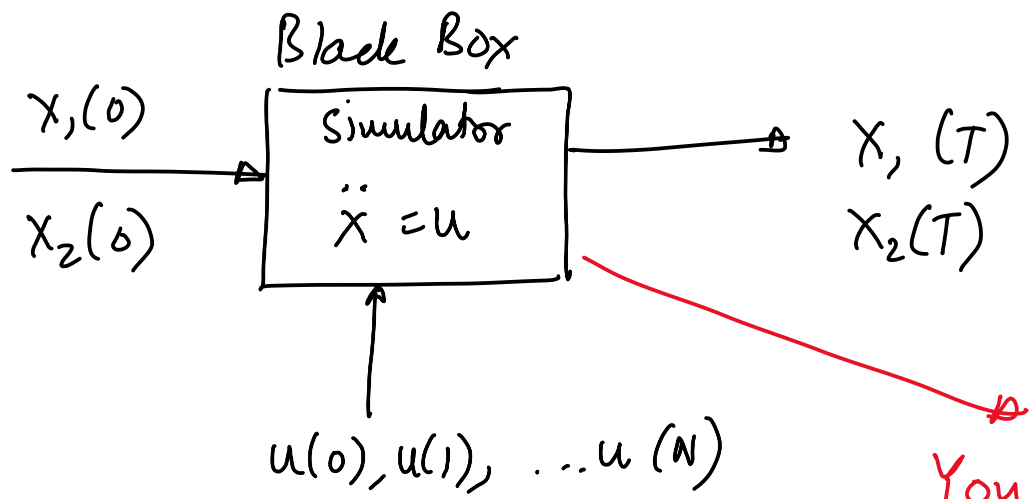
Thus there are $(3N+4) - (2N+4) = N$ free optimization variables

→ Hence, there are infinitely may ways of satisfying the constraints.

→ However, there only one solution that minimizes the cost (= time)

b) Shooting method

— Treats the dynamics as a black box (e.g. simulator)



You DONT have access to the equations because they are part of the Simulator

① optimization variable

$$T, \underbrace{u[0], u[1], \ldots u[N]}_{(N+1)}$$

T: 1

$N+2$

② Cost : T

③ Constraints:

$x_1(0) = 0$

$x_2(0) = 0$

Pre-specified

$$\left.\begin{aligned} x_1(N) &= D \\ x_2(N) &= 0 \end{aligned}\right\} \text{ 2 constraints}$$