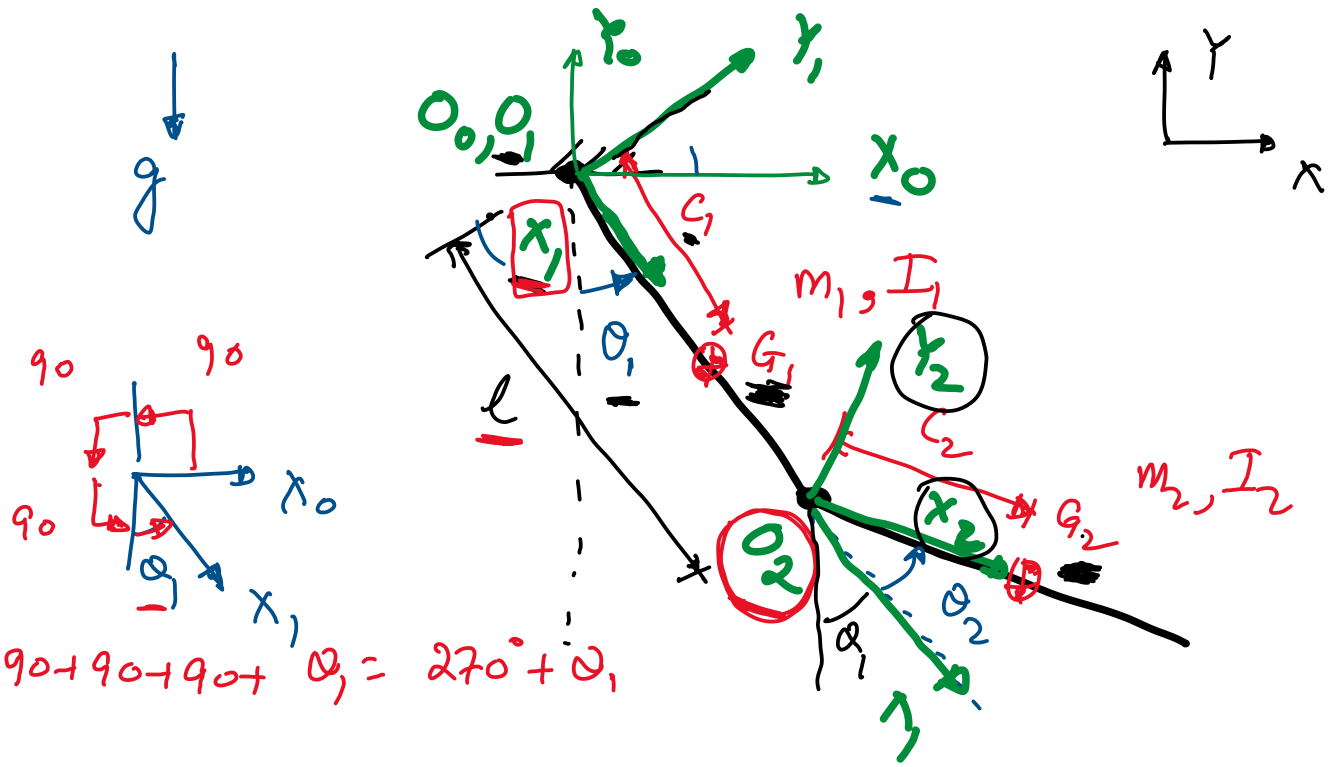
Double Pendulum: Derivation, simulate, animate



① Position of center of mass in from $O_0 - X_0 - Y_0$

$$G_1^0 = H_1^0 \, G_1^1$$

$$H_1^0 = \left[\begin{array}{c|c} R_1^0 & O_1^0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos(270+\theta_1) & -\sin(270+\theta_1) & 0 \\ \sin(270+\theta_1) & \cos(270+\theta_1) & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$G_1^1 = \begin{bmatrix} c_1 \\ 0 \\ 1 \end{bmatrix}$$

c_1 → distance along X_1

0 → distance along Y_1

1 → place holder to make computations possible.

$$G_1^0 = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix}$$

$$\rightarrow G_{T_1} = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \\ \underline{1} \end{bmatrix} \longleftarrow y^{0}_{G_1}$$

$$G_2^0 = H_2^0 \, G_2^2$$

$$G_2^0 = H_2^1 \, H_1^0 \, G_2^2$$

$$H_2^1 = \left[\begin{array}{c|c} R_2^1 & O_2^1 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_2 & -\sin\theta_2 & l \\ \sin\theta_2 & \cos\theta_2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$G_2^2 = \begin{bmatrix} c_2 \\ 0 \\ 1 \end{bmatrix}$$

$c_2 \rightarrow$ distance along x_2

$0 \rightarrow$ distance along y_2

$1 \rightarrow$ place holder

$$G_2^0 = \begin{bmatrix} l \sin\theta_1 + c_2 \sin(\theta_1 + \theta_2) \\ -l\cos\theta_1 - c_2 \cos(\theta_1 + \theta_2) \\ 1 \end{bmatrix}$$

$-l\cos\theta_1 - c_2 \cos(\theta_1 + \theta_2)$ — y_{G_2}

$$V_{G_1}^0 = \dot{G}_1^0 = \frac{d}{dt}\begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix} = \begin{bmatrix} c_1 \cos\theta_1 \, \dot{\theta}_1 \\ c_1 \sin\theta_1 \, \dot{\theta}_1 \end{bmatrix} \begin{matrix} \checkmark \\ \checkmark \end{matrix}$$

$$V_{G_2}^0 = \dot{G}_2^0 = \frac{d}{dt}\begin{bmatrix} l \sin\theta_1 + c_2 \sin(\theta_1+\theta_2) \\ -l \cos\theta_1 - c_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{(c_2 \cos(\theta_1+\theta_2) + l \cos\theta_1)\dot{\theta}_1 + c_2 \cos(\theta_1+\theta_2)\dot{\theta}_2}^{V_{G_2x}} \\ \underbrace{(c_2 \sin(\theta_1+\theta_2) + l \sin\theta_1)\dot{\theta}_1 + c_2 \sin(\theta_1+\theta_2)\dot{\theta}_2}_{V_{G_2y}} \end{bmatrix}$$

$$\dot{\theta}_1 = \omega_1 \quad ; \quad \dot{\theta}_2 = \omega_2$$

$$T = \frac{1}{2} m_1 V_{G_1}^2 + \frac{1}{2} m_2 V_{G_2}^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \underline{(\dot{\theta}_1 + \dot{\theta}_2)}^2$$

$$\left[\underbrace{(c_1 \cos\theta_1 \dot{\theta}_1)}_{x\text{-comp}}^2 + \underbrace{(c_1 \sin\theta_1 \dot{\theta}_1)}_{y\text{-component}}^2 \right]$$

$$\left(V_{G_2x}^2 + V_{G_2y}^2\right)$$

$$V = m_1 g \underbrace{y_{G_1}^0}_{-c_1 \cos\theta_1} + m_2 g \underbrace{y_{G_2}^0}_{-l\cos\theta_1 - c_2 \cos(\theta_1+\theta_2)}$$

$-c_1 \cos\theta_1$ $-l\cos\theta_1 - c_2 \cos(\theta_1+\theta_2)$

$\mathcal{L} = T - V$ ✓

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③ Euler-Lagrange Equations

$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \quad (q_j = \theta_1, \theta_2)$

$Q_j = 0$

$$\underline{A_{11}\ddot{\theta}_1 + A_{12}\ddot{\theta}_2} = b_1$$

$$A_{21}\ddot{\theta}_1 + A_{22}\ddot{\theta}_2 = b_2$$

A's, b's are functions of m, c, g, l, I

$$A\ddot{\theta} = b$$

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\ddot{\theta} = \underline{A}^{-1} b$$