Symbolic calculations

Motivation

Can we automate the computation of

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

This calculation becomes complex for even simple dynamical systems (e.g. double pendulum)

Symbolic derivatives

Hand calculations

$$f_0 = x^2 + 2x + 1$$

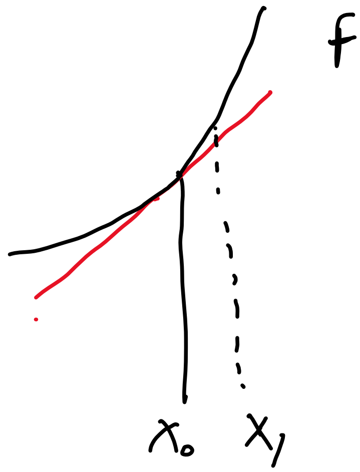
$$\frac{df_0}{dx} = 2x + 2$$

$$\frac{df_0}{dx}(x=1) = 2(1) + 2 = 4$$

Python

```
import sympy as sy

x = sy.symbols('x', Real="True")

f0 = x**2 + 2*x + 1

df0dx = sy.diff(f0, x)

df0dx.subs(x, 1)
```

Numerical derivative



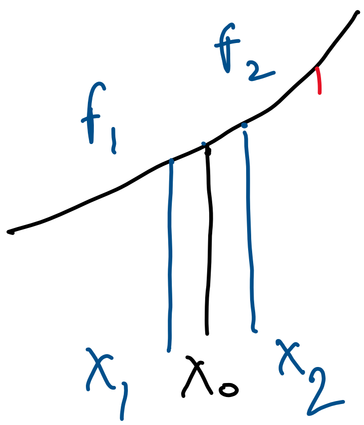
$$\boxed{\frac{df}{dx} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

x_1 is close to x_0

Forward Difference

$$x_1 = x_0 + 10^{-4}$$

(1e-4)



$$\boxed{\frac{df_0}{dx} = \frac{f_2 - f_1}{x_2 - x_1}}$$

central difference
It is more accurate than Forward Difference

$$x_1 = x_0 - 10^{-4}$$
$$x_2 = x_0 + 10^{-4}$$

Chain rule

If $f_1(x(t))$, compute $\dfrac{df_1}{dt}$

$$f_1 \longrightarrow x \longrightarrow t$$

$$\frac{df_1}{dt} = \underline{\frac{df_1}{dx}\ \frac{dx}{dt}}$$

Example

$$f_1 = \sin(x(t))$$

$$\frac{df_1}{dt} = \frac{d\left[\sin x(t)\right]}{dx}\ \frac{dx}{dt}$$

$$\boxed{\frac{df_1}{dt} = \cos(x(t))\ \dot{x}}$$

If $f_2(x(t), \dot{x}(t))$, then compute $\frac{df_2}{dt}$

position   velocity

$$f_2 \to x, \dot{x} \to t$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

EXAMPLE:

$$f_2 = x(t)\,\dot{x}(t)$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$$= \frac{d}{dx}(x\dot{x})\underbrace{\frac{dx}{dt}}_{\dot{x}} + \frac{d}{d\dot{x}}(x\dot{x})\frac{d\dot{x}}{dt}$$

$$= (\dot{x})\frac{dx}{dt} + x\frac{d\dot{x}}{dt}$$

$$\boxed{\frac{df_2}{dt} = \dot{x}^2 + x\ddot{x}}$$

Back to Euler-Lagrange for projectile

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$q_j = x, y$; $Q_j = F_{d_x}, F_{d_y}$ (drag force)

$$\mathcal{L} \longrightarrow x, y, \dot{x}, \dot{y} \longrightarrow t$$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \checkmark$ No chain rule

$\frac{\partial \mathcal{L}}{\partial x} = \checkmark$ No chain rule

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}}}_{\mathcal{L}_{\dot{x}}}\Big) = \underbrace{\frac{\partial \mathcal{L}_{\dot{x}}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \frac{\partial \mathcal{L}_{\dot{x}}}{\partial \dot{y}}\frac{d\dot{y}}{dt}}_{\text{chain rule}}$$