2D Dynamics

Newton's law 2nd law

$F = ma$ $\quad$ a = linear acceleration

$T = I\alpha$ $\quad$ α = angular acceleration

✓ ① Free Body Diagram

✓ ② Use Newton's laws

$F = ma$ or/and $T = I\alpha$

③ Given F, m, T, I solve for

a, α ~ Equations of motion

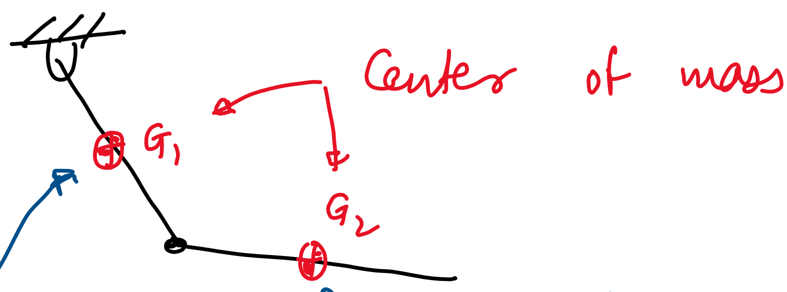
Euler-Lagrange Method

- Get equations of motion without drawing Free Body Diagram.

- We will also integrate the equations of motion and create animations.

Procedure for deriving Equations of motion using Euler-Lagrange Equations

① Write the positions of the center of mass with respect to the base /word frame



$$r^0_{G_1} = \begin{bmatrix} x^0_{G_1} \\ y^0_{G_1} \end{bmatrix} \qquad r^0_{G_2} = \begin{bmatrix} x^0_{G_2} \\ y^0_{G_2} \end{bmatrix}$$

② $\mathcal{L} = T - V$

$\mathcal{L}$ — Lagrangian

T — Kinetic Energy

V — Potential energy

$$T = \frac{1}{2} \sum_{i=1}^{n} \left(m_i v_i^2 + I_i \omega_i^2 \right)$$

v_i - linear speed

ω_i - angular speed

m_i - mass

I_i - Inertia

Differentiating the position in the world frame with respect to time

$$V = \sum_{i=1}^{n} m_i g_i y^0_{G_i} + \frac{1}{2}\sum_{i=1}^{q} k_{p_i}\left(r_{p_i} - r_{p_0}\right)^2$$

g_i = gravity

$y^0_{G_i}$ - position of center of mass

$\downarrow g$

$\uparrow y^0$

r_{p_i}, r_{p_0} — spring length, spring length in rest configuration

k_{p_i} — spring constant

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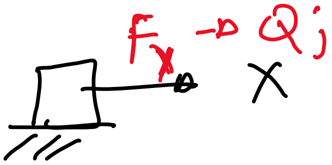
③ Equations of motion

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j}$$

→ gives same equations as those obtained using Newton's law

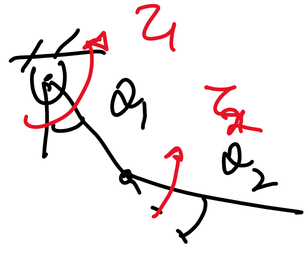
q_j — degrees of freedom

e.g.



$q_j = x$

$Q_j = F$

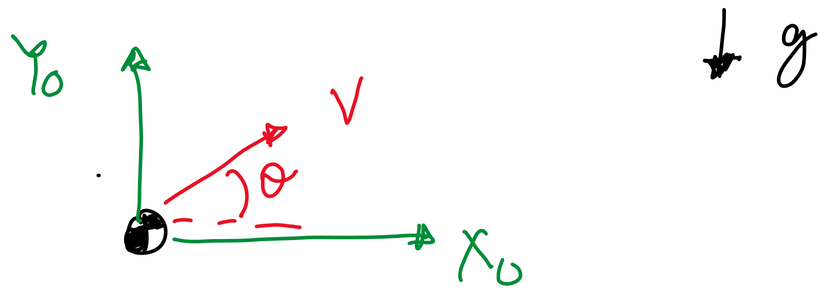


$q_j = \theta_1, \theta_2$

$Q_j = \tau_1, \tau_2$

Q_j — external forces not accounted in T or V

EXAMPLE: Projectile motion under quadratic drag force



Derive the equations of motion

Quadratic Drag

$$F_d = -\underline{c V^2}\ \hat{v}$$

V^2 – magnitude of drag force

$\hat{v}$ – unit vector

$$= \frac{\vec{V}}{|\vec{V}|}$$

$\hat{i}, \hat{j}$ are unit vectors in x-, y- direction

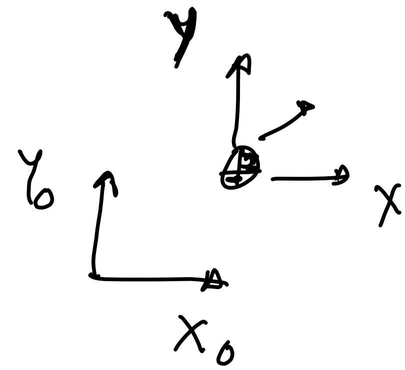
c = constant

$$\boxed{\begin{aligned} \vec{v} &= \dot{x}\,\hat{i} + \dot{y}\,\hat{j} \\ |\vec{v}| &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ V^2 &= \dot{x}^2 + \dot{y}^2 \\ \hat{v} &= \frac{\dot{x}\,\hat{i} + \dot{y}\,\hat{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \end{aligned}}$$

$$\boxed{F_d = -\underbrace{c\sqrt{\dot{x}^2 + \dot{y}^2}\ \dot{x}}_{F_{dx}}\ \hat{i} - \underbrace{c\sqrt{\dot{x}^2 + \dot{y}^2}\ \dot{y}}_{F_{dy}}\ \hat{j}}$$

Euler - Lagrange equations

① x, y



② $\mathcal{L} = T - V$

$$T = \frac{1}{2} m v^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

$$\mathcal{L} = T - V = \frac{1}{2} m \underline{(\dot{x}^2 + \dot{y}^2) - mgy}$$

③ $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$q_j = [x, y]$; $Q_j = [F_{dx}, F_{dy}]$

$q_j = x$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left\{ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy \right\}$$

$$= m\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt} m\dot{x} = \underline{\underline{m\ddot{x}}}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}\left\{ \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy \right\} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_{dx}$$

$$\boxed{m\ddot{x} - 0 = -c\sqrt{\dot{x}^2+\dot{y}^2}\;\dot{x}}$$

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)} - \underbrace{\frac{\partial \mathcal{L}}{\partial y}} = F_{dy}$$

$$\underbrace{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{y}}\left\{\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy\right\}\right]}$$

$$- \underbrace{\frac{\partial}{\partial y}\left[\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy\right]} = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}$$

$$\frac{d}{dt}\left[\frac{1}{\cancel{2}} m (\cancel{2}\dot{y})\right] + mg = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}$$

$$\boxed{m\ddot{y} + mg = -c\sqrt{\dot{x}^2 + \dot{y}^2}\;\hat{y}}$$

Summary

$$\boxed{\begin{aligned} \ddot{x} &= -\frac{c}{m}\,\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} \\ \ddot{y} &= -g - \frac{c}{m}\,\dot{y}\sqrt{\dot{x}^2+\dot{y}^2} \end{aligned}}$$

(4) Simulate and animate in python

Car : $\dot{x} = v\cos\theta$; $\dot{y} = v\sin\theta$; $\dot{\theta} = \omega$

$$\underbrace{x_{t+1} = x_t + h\, v\cos\theta}_{\text{Euler's integration}} \quad \text{\& so on}$$

To compute x, y

h = constant

(1) Euler's integration

(2) Runge-Kutta method (RK4)

h variable

(3) $\underbrace{\text{Adaptive Runge-Kutta}}$ method — odeint

output: $x, y, \dot{x}, \dot{y}$

$$z = \text{odeint}(\underbrace{\text{projectile\_rhs}}_{\ddot{x},\ \ddot{y},\ \dot{x},\ \dot{y}},\ \underbrace{z_0}_{\text{init condition}},\ \underbrace{t}_{\text{time}},\ \underbrace{\text{arguments}}_{m,\ c,})$$