# Jacobian

Say $\mathbf{f} = [f_1(q), f_2(q), f_3(q), \ldots f_m(q)]$

↑ vector of functions

$\mathbf{q} = [x_1, x_2, \ldots \quad x_n]$

$$J = \frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$\underline{m \times n}$

$$f = [\overbrace{x^2+y^2}^{f_1}, \underline{\overbrace{2x+3y+5}^{f_2}}]$$

$$q = [x, y]$$

$$\frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2 & 3 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial q}\right|_{(1,2)} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$$

# Applications of the Jacobian (3)

## ① : Cartesian velocity

$\dot{x}, \dot{y}, \dot{z}$

**Theory**

$$p^0 = f(q)$$

joint angles

position of point $p$ in frame 0

$$J = \frac{\partial f}{\partial q} \Rightarrow \partial f = J \partial q$$

$$\frac{df}{dt} = J \frac{dq}{dt}$$

$p^0 = f(q)$

$$\frac{dp^0}{dt} = J \frac{dq}{dt}$$

$$\dot{p}^0 = J \dot{q}$$

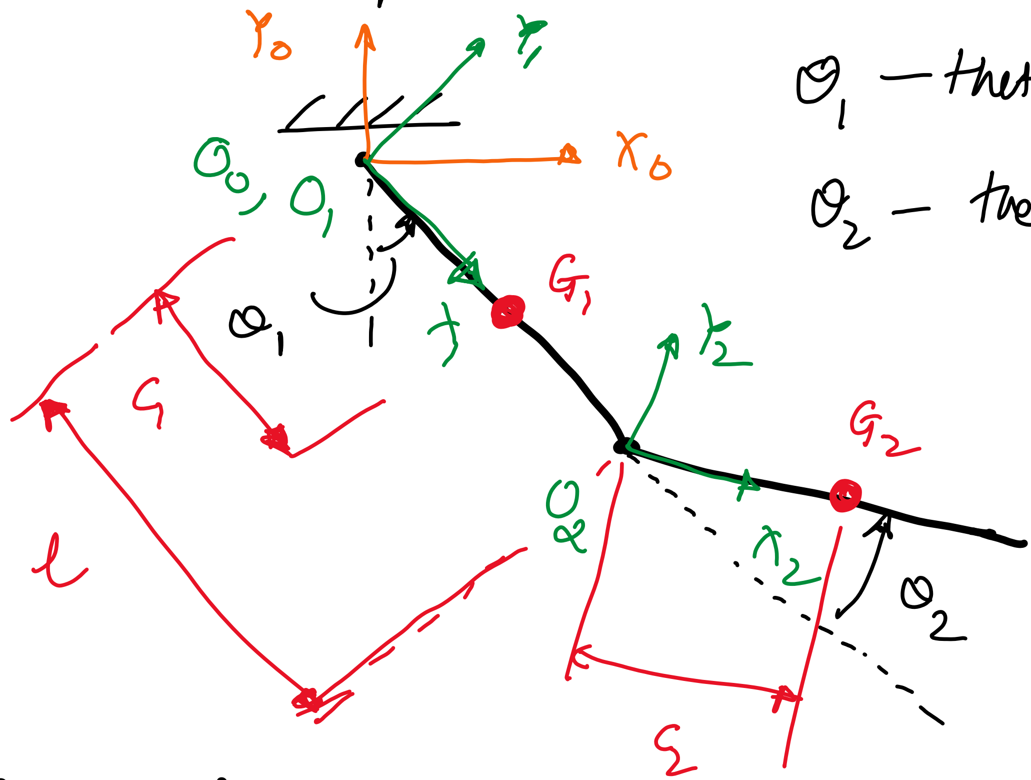
$$\begin{bmatrix} \dot{x}_p^0 \\ \dot{y}_p^0 \end{bmatrix} = J \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \vdots \end{bmatrix}$$

$$\boxed{v_p^0 = J \dot{q}}$$

cartesian velocity

angular velocity

Example : Velocity of center of mass of a double pendulum




$\theta_1$ — theta 1

$\theta_2$ — theta 2

$V^0_{G_1} = ?$ $\quad V^0_{G_2} = ?$ $\quad$ Given $\dot\theta_1, \dot\theta_2$ (angular velocity)

$$V^0_{G_1} = \boxed{J_{G_1}}\, \dot q$$

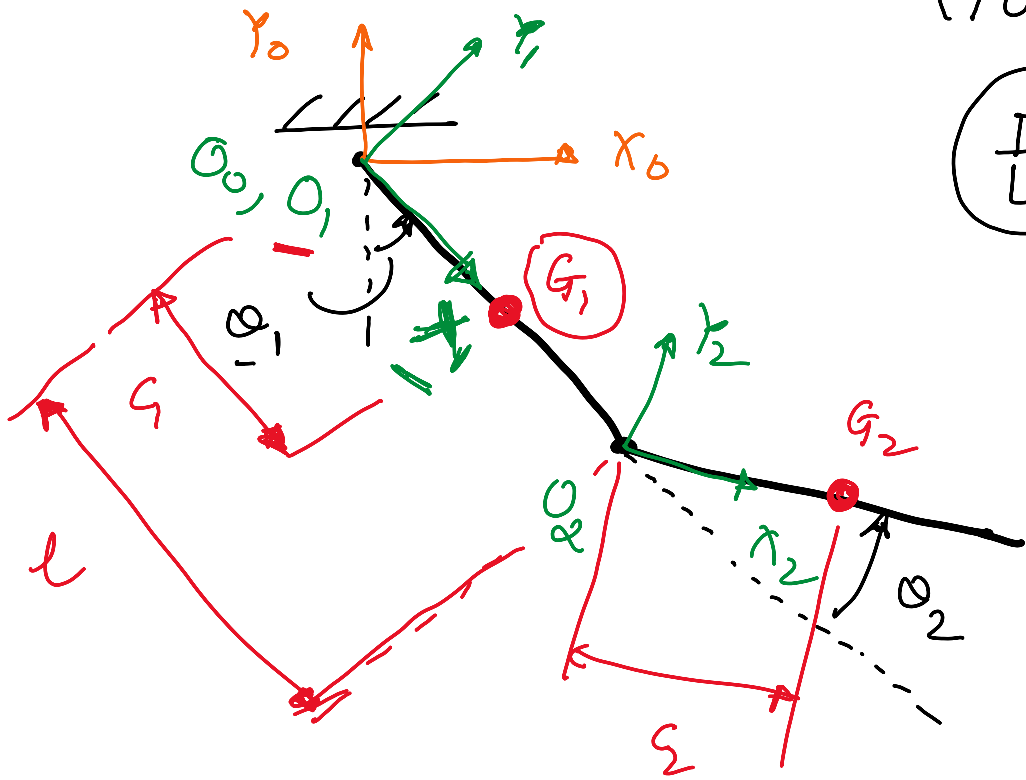
$$= J_{G_1} \underbrace{\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}}_{\text{angular velocity}}$$

$$V^0_{G_2} = \boxed{J_{G_2}}\, \dot q$$

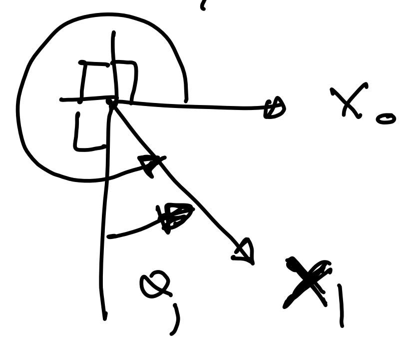
$$= J_{G_2} \underbrace{\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}}_{\text{angular velocity}}$$

$$J_{G_1} = \frac{\partial f_{G_1}}{\partial q} = \frac{\partial g_1^0}{\partial q}$$

$$G_1^0 = H_1^0 \, G_1^1$$




$270^\circ + \theta_1$




$$H_1^0 = \begin{array}{c} \begin{array}{cc} R_1^0 & o_1^0 \end{array} \\ \left[\begin{array}{cc|c} \cos(270^\circ+\theta_1) & -\sin(270^\circ+\theta_1) & 0 \\ \sin(270^\circ+\theta_1) & \cos(270^\circ+\theta_1) & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \end{array}$$

$$G_1^1 = \begin{bmatrix} c_1 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \text{distance along } X_1 \\ \text{distance along } Y_1 \\ \phantom{x} \end{array}$$

$$G_1^0 = \overbrace{\begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{H_1^0} \overbrace{\begin{bmatrix} c_1 \\ 0 \\ 1 \end{bmatrix}}^{G_1^1}$$

$$= \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \\ 1 \end{bmatrix} \Big\} \, g_1^0$$

$$g_1^0 = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix} = \begin{bmatrix} x_{G_1}^0 \\ y_{G_1}^0 \end{bmatrix} \qquad q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$J_{G_1} = \frac{\partial g_1^0}{\partial q} = \begin{bmatrix} \dfrac{\partial x_{G_1}^0}{\partial \theta_1} & \dfrac{\partial x_{G_1}^0}{\partial \theta_2} \\ \dfrac{\partial y_{G_1}^0}{\partial \theta_1} & \dfrac{\partial y_{G_1}}{\partial \theta_2} \end{bmatrix}_{2\times 2} \leftarrow$$

$$J_{G_1} = \begin{bmatrix} (c_1 \cos\theta_1) & 0 \\ (c_1 \sin\theta_1) & 0 \end{bmatrix}_{2\times 2}$$

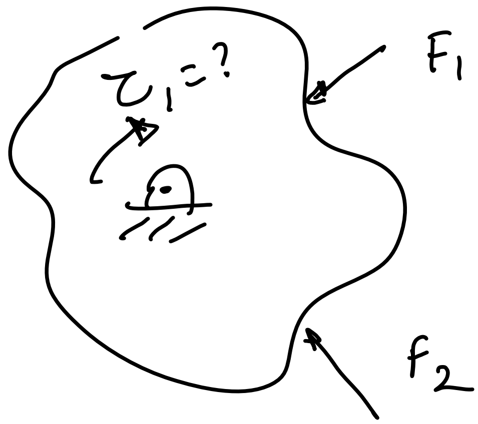
$$V_{G_1} = J_{G_1} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \Rightarrow \boxed{V_{G_1} = \begin{bmatrix} c_1 \cos\theta_1 \, \dot{\theta}_1 \\ c_1 \sin\theta_1 \, \dot{\theta}_1 \end{bmatrix}}_{2\times 1}$$

Do this computation at home

$$V^{0}_{G_2} = \underbrace{\begin{bmatrix} c_2 \cos(\theta_1+\theta_2) + l\cos\theta_1 & c_2\cos(\theta_1+\theta_2) \\ c_2 \sin(\theta_1+\theta_2) + l\sin\theta_1 & c_2 \sin(\theta_1+\theta_2) \end{bmatrix}}_{J_{G_2}} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

$2\times 2$ $\quad$ $2\times 1$

## ② Computing static forces




Given $F$'s, compute $\tau_1$ to keep the object in static equilibrium

### Theory

#### Virtual Work

$$\underbrace{\text{Work}}_{1\times1} = \underbrace{F^T}_{1\times2}\,\underbrace{\delta r}_{2\times1}$$

$\delta r$ ← virtual displacement

equate

$$\underbrace{\text{Work}}_{1\times1} = \underbrace{\tau^T}_{1\times1}\,\underbrace{\delta\theta}_{1\times1}$$

$$\tau^T \delta \alpha = F^T \delta r$$

$$\tau^T = F^T \frac{\delta r}{\delta \theta} \qquad \frac{\partial f}{\partial q} = J$$

$$\tau^T = \underline{\underline{F^T J}}$$

Taking transpose of both sides

$$\tau = (F^T J)^T$$

$$= J^T (F^T)^T \qquad \{(AB)^T = B^T A^T\}$$

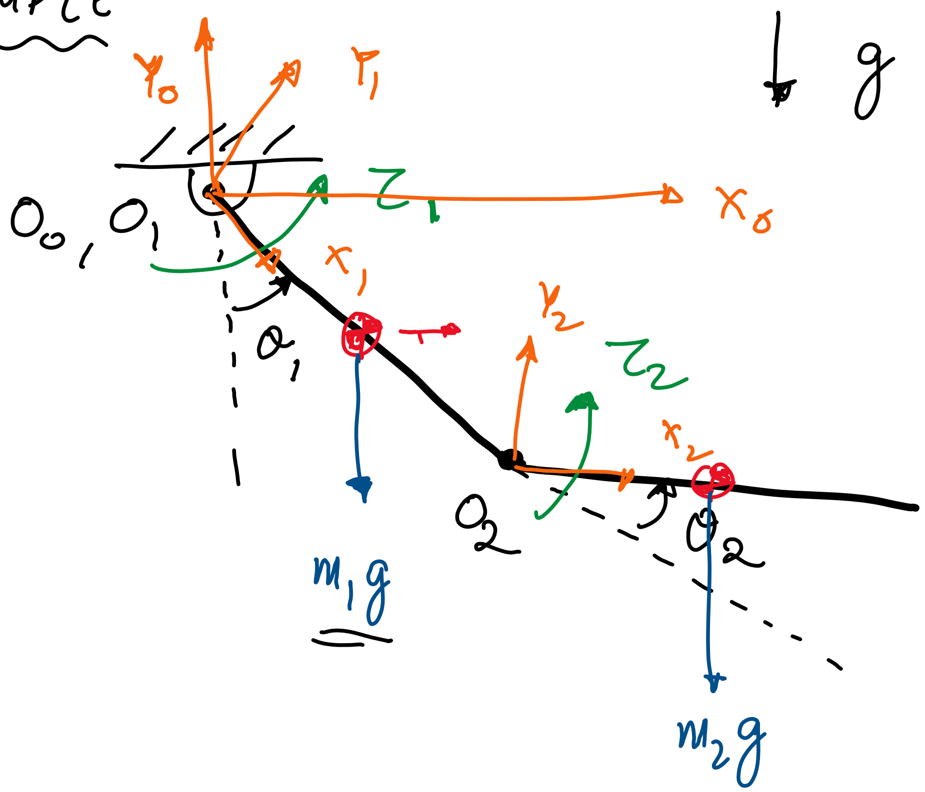
$$\boxed{\tau = J^T F}$$

moment arm

$$V = J \dot{q}$$

$$V = \underline{\underline{\omega \times r}}$$

# EXAMPLE




Compute $\tau_1$ and $\tau_2$ such that the pendulum is in static equilibrium $\theta_1 \neq 0, \theta_2 \neq 0$

$$\tau = \sum J^T F$$

$$F^0_{g_1} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underset{\sim}{J}_{G_1}^T \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + \underset{\sim}{J}_{G_2}^T \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

This was computed earlier.

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 \cos\theta_1 & c_1 \sin\theta_1 \\ 0 & 0 \end{bmatrix}}_{J_{G_1}^T} \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + \cdots$$

$$\underbrace{2\times 2 \quad 2\times 1}_{2\times 1}$$

$$\underbrace{\begin{bmatrix} c_2 \cos(\theta_1+\theta_2) + l \cos\theta_1 & c_2 \sin(\theta_1+\theta_2) + l \sin\theta_1 \\ c_2 \cos(\theta_1+\theta_2) & c_2 \sin(\theta_1+\theta_2) \end{bmatrix}}_{J_{G_2}^T} \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -m_1 g c_1 \sin\theta_1 - m_2 g c_2 \sin(\theta_1+\theta_2) - m_2 g l \sin\theta_1 \\ -m_2 g c_2 \sin(\theta_1+\theta_2) \end{bmatrix}}$$

## ③ Application of Jacobian: Computing inverse kinematics

So far, we have used fsolve to compute inverse kinematics.

Theory $\quad v = J \dot{q}$ [See application 1: Computing linear velocity]

$v$: Cartesian velocity

$\dot{q}$: joint velocity

$$\frac{dr}{dt} = J \frac{dq}{dt}$$

$r = [x, y]$

$q = [\theta_1, \theta_2, \ldots ]$

$$dr = J \, dq$$

$$J^{-1} dr = dq$$

$$\Rightarrow \quad dq = J^{-1} dr$$

$$dq = J^{-1} [ r_{ref} - r ]$$

$dq$: update in joint

$r_{ref} = \begin{bmatrix} x_{ref} \\ y_{ref} \end{bmatrix}$: reference motion

$r = \begin{bmatrix} x \\ y \end{bmatrix}$: actual position reference motion — actual position

$$\boxed{dq} = J^{-1}[r_{ref} - r]$$

update in joint

$\begin{bmatrix} x_{ref} \\ y_{ref} \end{bmatrix}$ reference motion

$\begin{bmatrix} x \\ y \end{bmatrix}$ actual position

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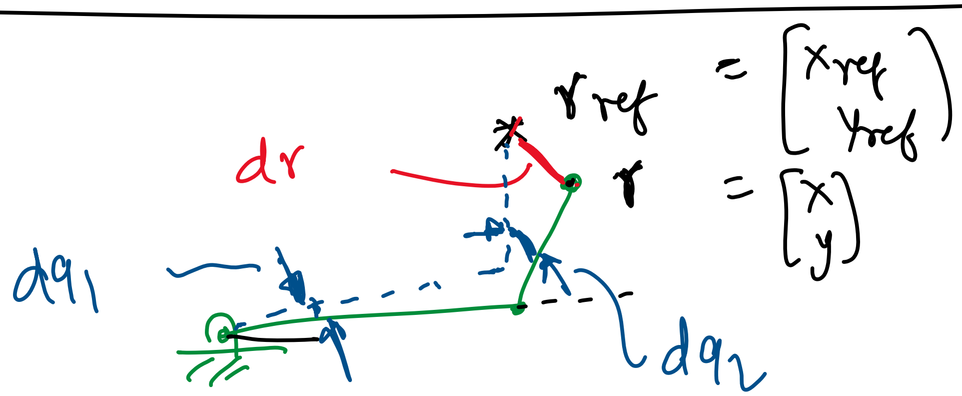


$$dq = J^{-1} dr$$

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix}$$

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$$\underset{new}{\theta_1} = \underset{old}{\theta_1} + dq_1$$

$$\underset{new}{\theta_2} = \underset{old}{\theta_2} + dq_2$$