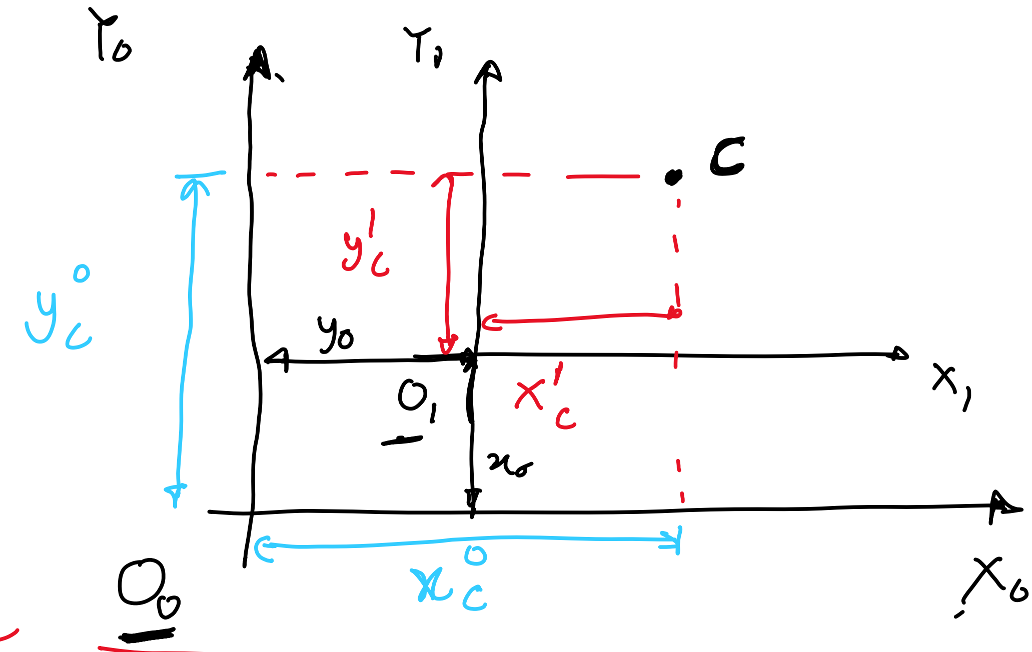
Coordinate Frames: Translation & Rotation

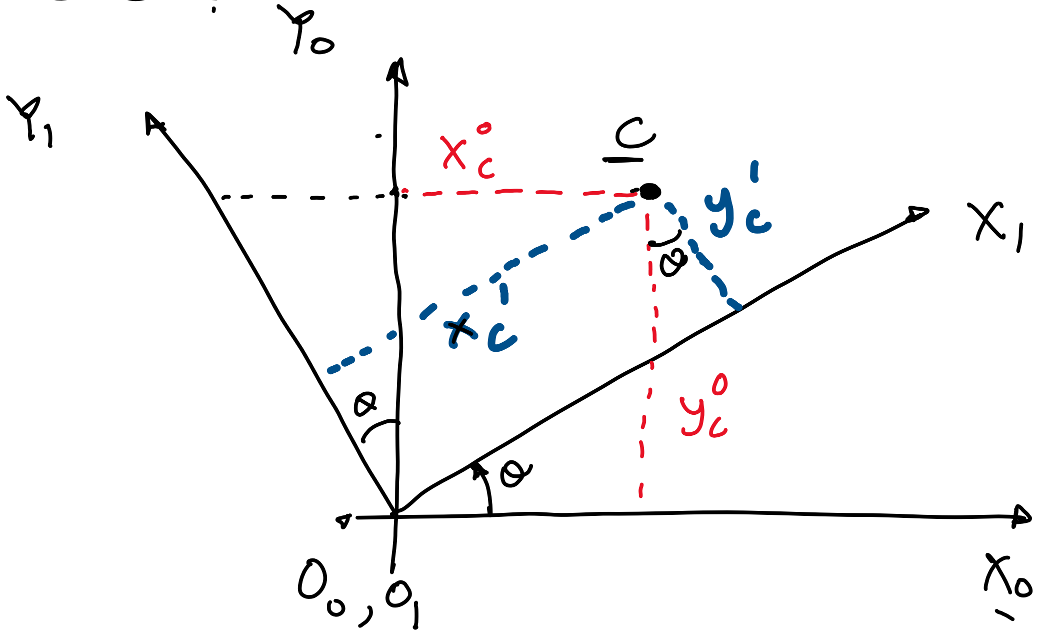
1.1 Translation



$$c^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} \qquad c^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$

$$O_1^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \; ; \; O_0^1 = \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix}$$

1.2 Rotation



$$x_c^0 = x_c^1 \cos\theta - y_c^1 \sin\theta$$

$$y_c^0 = x_c^1 \sin\theta + y_c^1 \cos\theta$$

$$\underbrace{\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} x_c^1 \\ \underline{y_c^1} \end{bmatrix}$$

Corrected

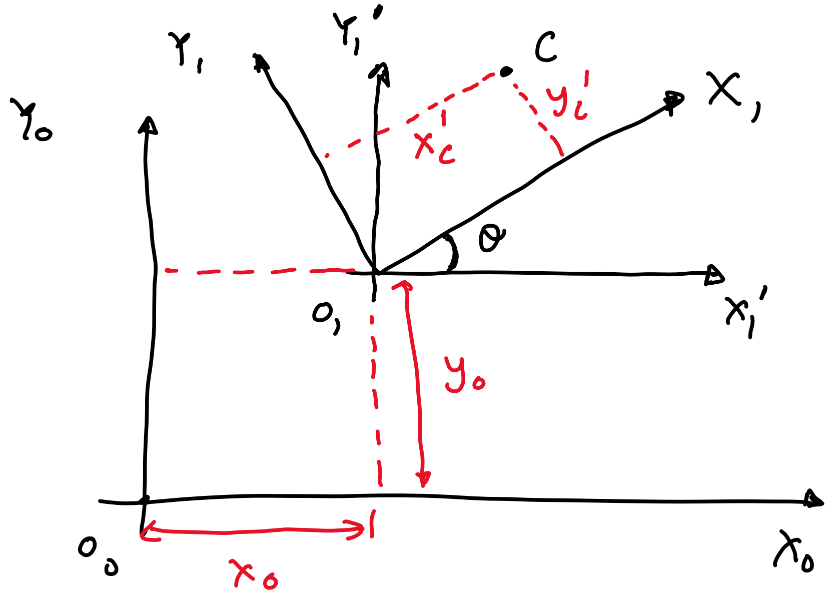
$$\boxed{\begin{aligned} c^0 &= R_1^0 c^1 \\ c^1 &= (R_1^0)^T c^0 \end{aligned}}$$

$$\underline{c}^0 = R_1^0 \underline{c}^1$$

$$c^1 = [R_1^0]^{-1} c^0$$

$$c^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} c^0 = (R_1^0)^T c^0$$

1.3 Combined rotation and translation



$$\underline{\underline{O_0 X_0 Y_0}} \longrightarrow O_1 X_1' Y_1' \longrightarrow \underline{\underline{O_1 X_1 Y_1}}$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\substack{\text{Translation} \\ 1.1}} + \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c' \\ y_c' \end{bmatrix}}_{\substack{\text{Rotation} \\ 1.2}}$$

1.3 Translation + Rotation

$$\boxed{c^0 = O_1^0 + R_1^0 c^1}$$

1.4 Multiple successive translations and rotations

$O_0 X_0 Y_0 \longrightarrow O_1 X_1 Y_1$

$$c^0 = o_1^0 + R_1^0 c^1 \qquad -(1)$$

$O_1 X_1 Y_1 \longrightarrow O_2 X_2 Y_2$

$$c^1 = o_2^1 + R_2^1 c^2 \qquad -(2)$$

Substitute (2) in (1) to get

$$c^0 = o_1^0 + R_1^0 \left(o_2^1 + R_2^1 c^2 \right)$$

$$c^0 = o_1^0 + R_1^0 o_2^1 + R_1^0 R_2^1 c^2$$

$$c^0 = \underbrace{\left(o_1^0 + R_1^0 o_2^1 \right)}_{\text{translation}} + \underbrace{\left(R_1^0 R_2^1 \right)}_{\text{rotation part}} c_2$$

continue doing this

$O_0 X_0 Y_0 \rightarrow O_1 X_1 Y_1 \rightarrow O_2 X_2 Y_2 \rightarrow \ldots \rightarrow O_n X_n Y_n$

$$c^0 = \left(o_1^0 + \underbrace{R_1^0 o_2^1 + R_1^0 R_2^1 o_3^2 + \ldots}_{\text{translation}} R_1^0 R_2^1 R_3^2 \ldots R_{n-1}^{n-2} o_n^{n-1} \right) + \underbrace{R_1^0 R_2^1 R_3^2 R_4^3}_{\text{rotation}} \ldots R_n^{n-1} c^n$$

(III)

This is unwieldy → Better way.

1.5 Homogenous transformation — Better way to keep track of multiple frames

$$H_i^{i-1} = \underset{3\times 3}{\begin{bmatrix} \underset{2\times 2}{R_i^{i-1}} & \underset{2\times 1}{O_i^{i-1}} \\ \underset{1\times 2}{0} & \underset{1\times 1}{1} \end{bmatrix}}$$

$$c^i = \underset{3\times 1}{\begin{bmatrix} \underset{2\times 1}{c^i} \\ \underset{1\times 1}{1} \end{bmatrix}} \quad \text{one}$$

$$\underset{3\times 1}{c^{i-1}} = \underset{3\times 3}{H_i^{i-1}} \; \underset{3\times 1}{c^i}$$

$$\begin{bmatrix} c^{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c^i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c^{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} c^i + O_i^{i-1} \\ 0 + 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} c^i + O_i^{i-1} \\ 1 \end{bmatrix}$$

$$c^{i-1} = R_i^{i-1} c^i + O_i^{i-1}$$

e.g. $i = 0$ $\quad c^0 = R_1^0 c^1 + O_1^0 \rightsquigarrow O_0 X_0 Y_0 \to O_1 X_1 Y_1$

$i = 1$ $\quad c^1 = R_2^1 c^2 + O_2^1 \rightsquigarrow O_1 X_1 Y_1 \to O_2 X_2 Y_2$

$$\boxed{c^0 = H_1^0 H_2^1 H_3^2 \cdots H_n^{n-1} c^n}$$

— Simpler way of writing (III)

$\underline{C} = \Pi_1 \Pi_2 \Pi_3 \cdots \Pi_n \underline{C}$ of writing (III)