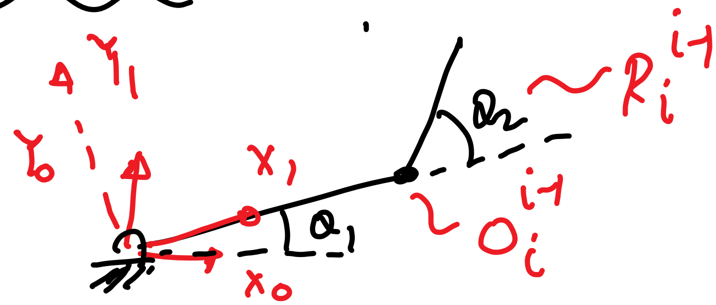
3D manipulators

2D

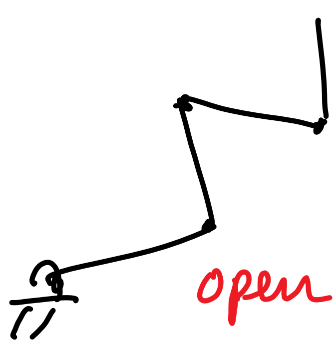


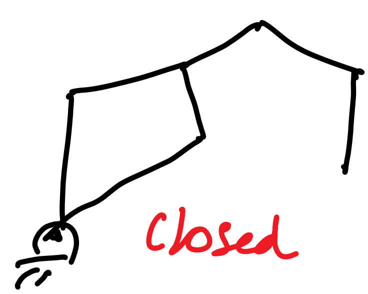
$$H_i^{i-1} = \begin{bmatrix} \underset{2\times 2}{R_i^{i-1}} & \underset{2\times 1}{O_i^{i-1}} \\ \underset{1\times 2}{(0\ 0)} & \underset{1\times 1}{1} \end{bmatrix}_{3\times 3}$$

3D

$$H_i^{i-1} = \begin{bmatrix} \underset{3\times 3}{R_i^{i-1}} & \underset{3\times 1}{O_i^{i-1}} \\ \underset{1\times 3}{(0\ 0\ 0)} & \underset{1\times 1}{1} \end{bmatrix}_{4\times 4}$$

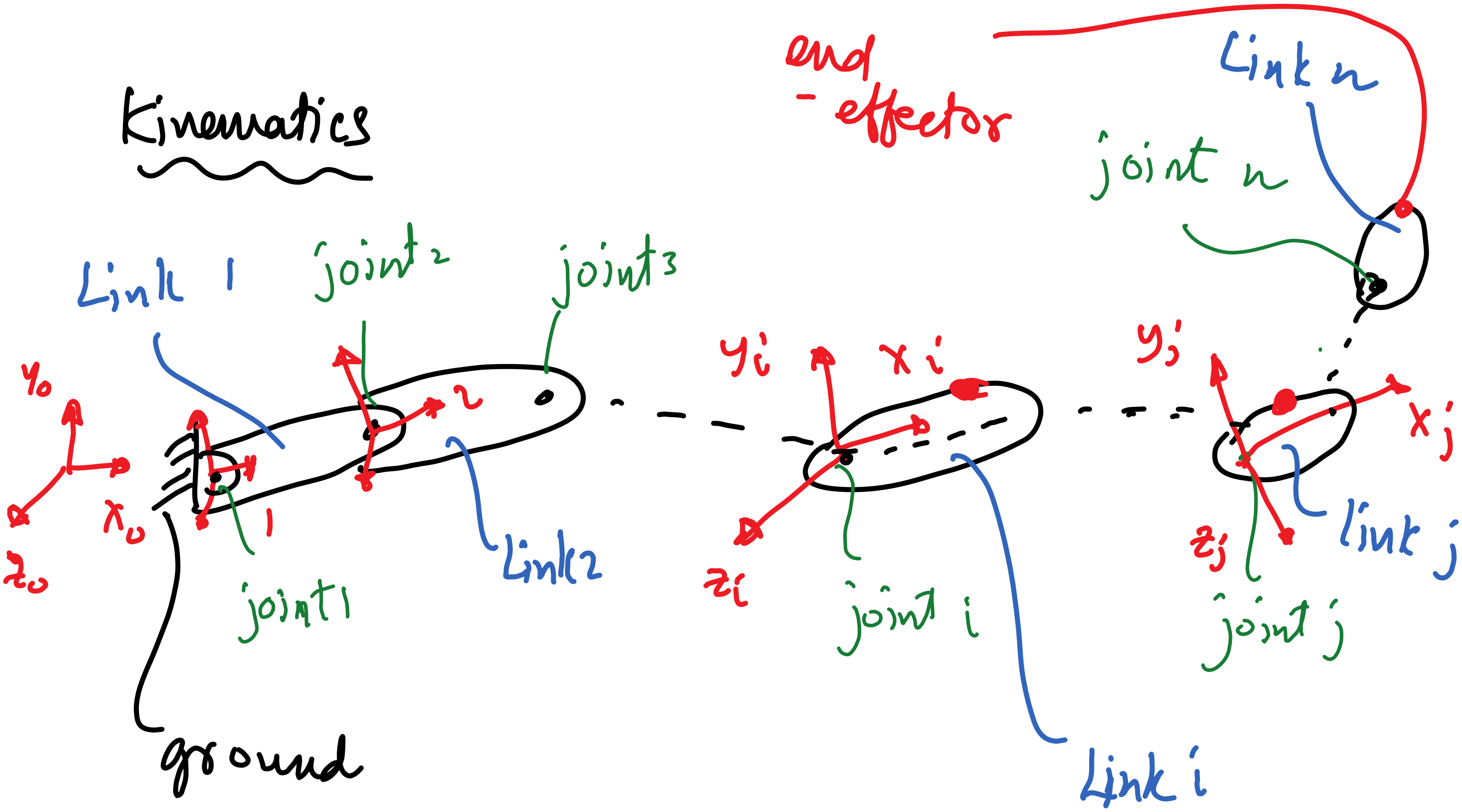
we will use this





(X) not in this course

Kinematics



$$H^i_j = H^i_{i+1}\, H^{i+1}_{i+2} \cdots H^{j-1}_j \qquad i<j$$

$$= I \sim \text{identity matrix} \qquad i=j$$

$$= \left(H^j_i\right)^{-1} \qquad i>j$$

Homogenous matrix (4×4)

$$H_x(\phi) = \begin{array}{cc} \qquad\qquad\qquad R & O \\ \left[\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{array}\right]_{4\times4} \end{array}$$

$$H_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

$$H_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Revolute motion}}$

$$H_x(a_x) = \begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad H_y(a_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_z(a_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_z \end{bmatrix}$$

$\boxed{\text{Prismatic motion}}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

motion

Denavit-Hartenberg convention (DH)

Efficient method to represent kinematics of a manipulator

$$H_i^{i-1} = H_z(\theta_i)\, H_z(d_i)\, H_x(a_i)\, H_x(\alpha_i)$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c\theta_i = \cos\theta_i$ $\qquad$ $c\alpha_i = \cos\alpha_i$

$s\theta_i = \sin\theta_i$ $\qquad$ $s\alpha_i = \sin\alpha_i$

$\theta_i, d_i, a_i, \alpha_i$ — Denavit Hartenberg (DH) parameters

Although 6 numbers $(x, y, z, \theta, \psi, \phi)$ are needed to describe a link position/orientation DH does this with only 4 because DH uses a special way to define the axis of the links axis of the links.

① axis of x_i is perpendicular to z_{i-1}

② axis of x_i is intersecting with z_{i-1}

These constraints help get rid of 2 parameters.

Algorithm for using DH for forward kinematics There are three steps.

1. **Assign coordinate frames:**
   (a) Assign z_i along the axis of actuation for each link, where $i = 0, 1, 2, ...(n-1)$.
   (b) Assign the base frame $o_0 - x_0 - y_0 - z_0$. The z_0 has already been assigned. Assign x_0 arbitrarily. Assign y_0 based on x_0 and z_0 using right hand rule.

   (c) Now assign coordinate frames $o_i - x_i - y_i - z_i$ for $i = 1, 2, ..., n-1$. z_i is already attached in first step. Next we assign x_i using these rules.
      i. **z_{i-1} and z_i are not coplanar:** In this case, there is a unique shorted distance segment that is perpendicular to z_{i-1} and z_i. Choose this as x_i axis. The origin o_i is where x_i intersects z_i. The y_i is found from right hand rules.
      ii. **z_{i-1} and z_i parallel:** In this case, there infinitely many perpendiculars. Choose any of these perpendiculars for x_i. Furthermore, where x_i intersects z_i we draw the origin x_i. Finally, y_i is found from the right hand rule. To make equations simpler, choose x_i such that is passes through o_{i-1}. This will make $d_i = 0$. Also, since z_{i-1} is parallel to z_i, $\alpha_i = 0$.
      iii. **z_{i-1} and z_i intersect:** In this case, x_i is chosen to be normal to the plane formed by z_{i-1} and z_i. There will be two possible directions for x_i, one of them is chosen arbitrarily and o_i is obtained by the intersection of $z - i$ and x_i. Finally y_i is obtained from right hand rule. Also, since z_{i-1} intersects z_i, $a_i = 0$.

   (d) Finally we need to attach an end effector frame, $o_n - x_n - y_n - z_n$. Attach z_n to be the same direction as z_{n-1}. Now depending on the relation between z_n and z_{n-1}, attach frame x_n. Finally, attach y_n using the right hand rule.

$n = 2$: z_1 & z_2

2. **Generate a table for DH parameter:** Now generate the DH table as follows.

Link	a_i	α_i	d_i	θ_i
1				
2				
.				
.				
.				
n				

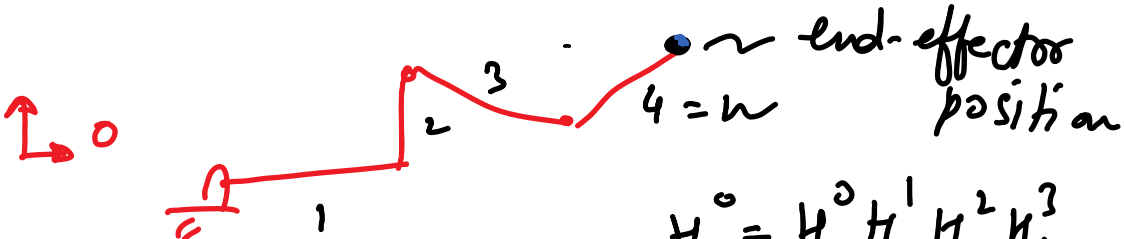
3. **Apply DH transformation to evaluate forward kinematics:** Finally, use the DH formulate to link two adjacent frames

$$\mathbf{H}_i^{i-1} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The position and orientation of the end-effector is found using the formula

$$\mathbf{H}_n^0 = \mathbf{H}_1^0 \mathbf{H}_2^1 \mathbf{H}_3^2 ... \mathbf{H}_n^{n-1} = \begin{bmatrix} \mathbf{R}_n^0 & \mathbf{d}_n^0 \\ \mathbf{0} & 1 \end{bmatrix}$$

The position of the end-effector is $\mathbf{d}_n^0$ and the orientation is $\mathbf{R}_n^0$. From $\mathbf{R}_n^0$, we can recover the Euler angles for the end-effector frame.



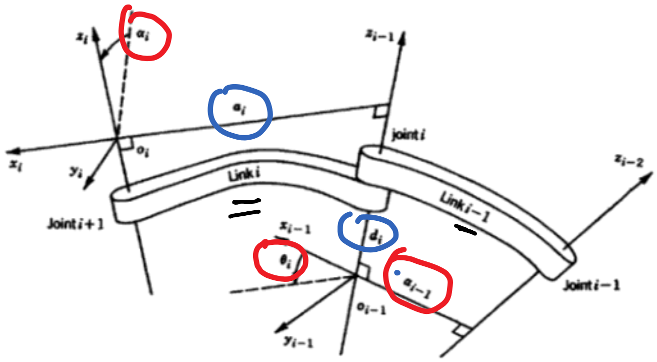
$$H_4^0 = H_1^0 H_2^1 H_3^2 H_4^3$$

$$= \begin{bmatrix} (R_n^0) & (d_n^0) \\ 0 & 1 \end{bmatrix}$$

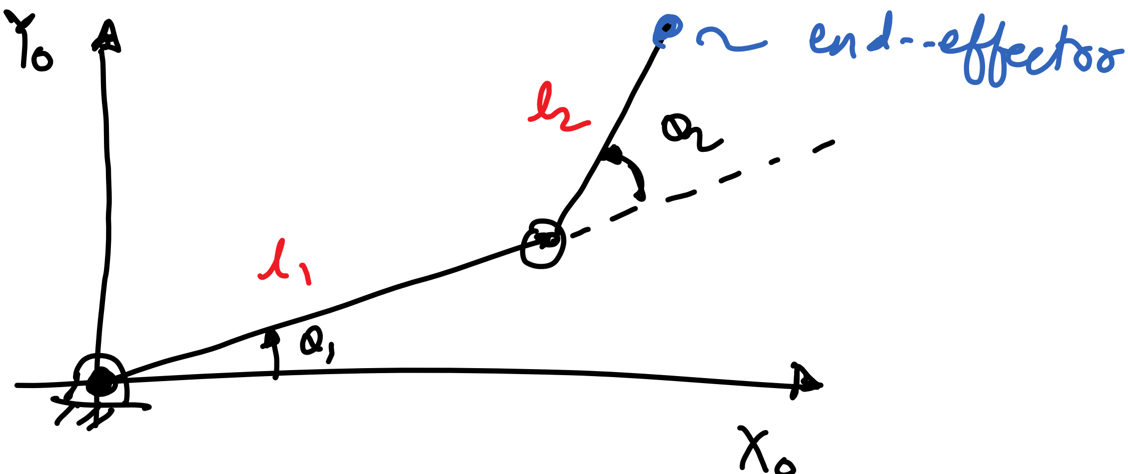
orientation of end-effector

position of end-effector

1. a_i is the distance between z_i and z_{i-1} along x_i.
2. α_i is the angle between z_i and z_{i-1} along x_i.
3. d_i is the distance between x_{i-1} and x_i along z_{i-1}.
4. θ_i is the angle between x_{i-1} and x_i along z_{i-1}.



1) Example 1: 2-D planar manipulator



Compute the position and orientation of the end-effector.

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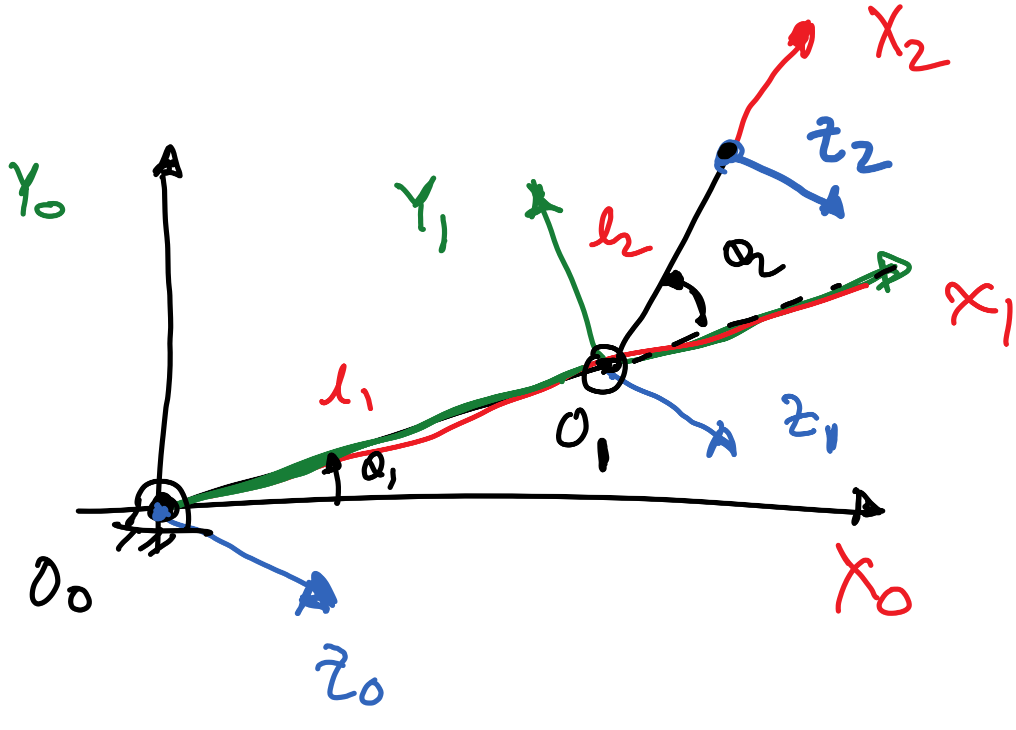
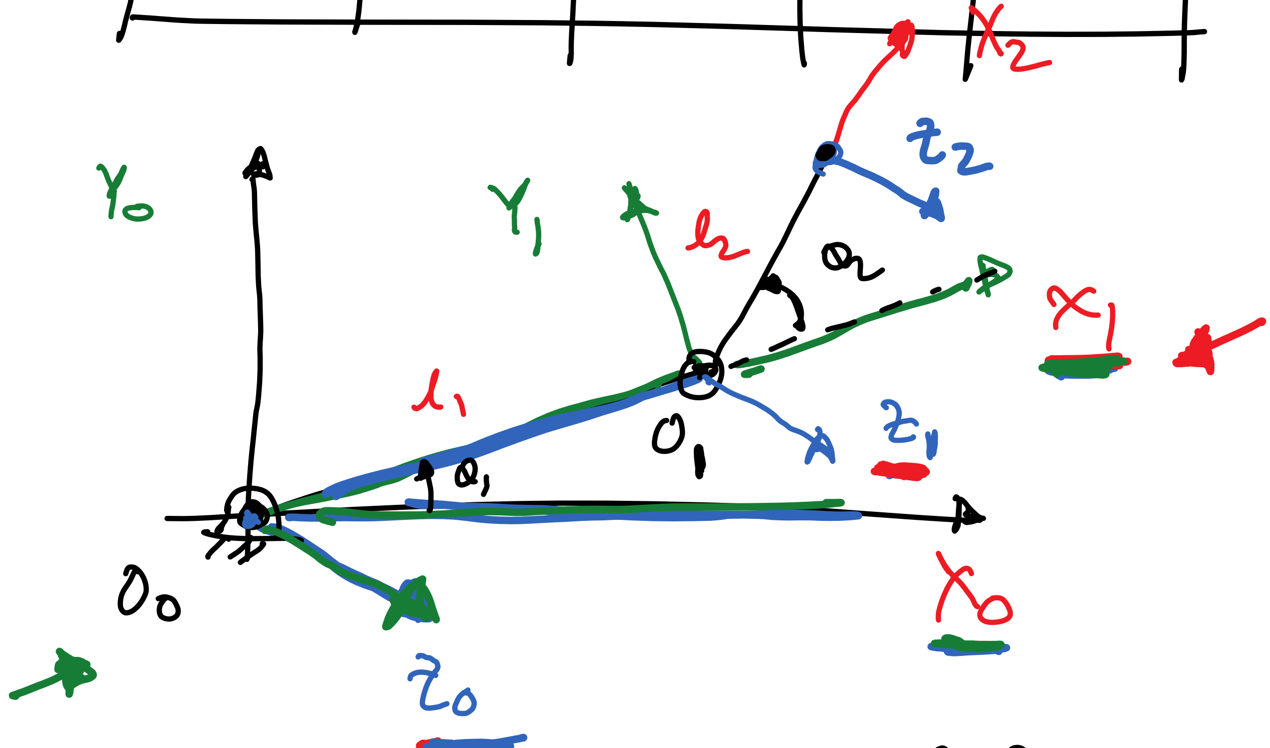
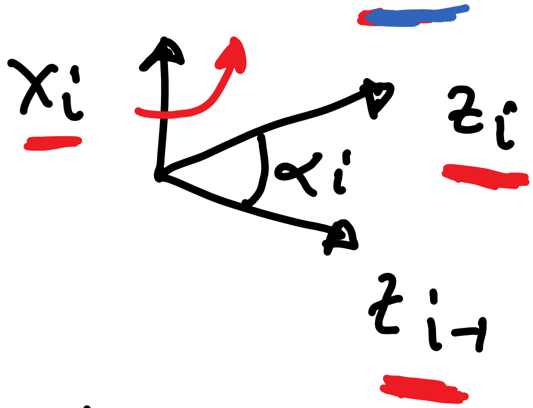
1) <u>Assign co-ordinate frames</u>

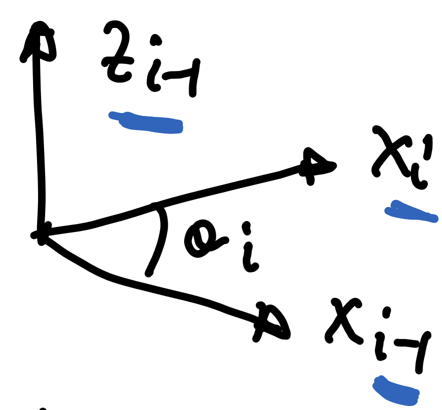


② Generate the DH table

Link	a_i	α_i	d_i	θ_i	
1	l_1	0	0	θ_1	
2	l_2	0	0	θ_2	— same reasoning







a_i, α_i

(z_0, z_1) x_1

d_i, θ_i

(x_0, x_1) z_0

(z_0, z_1) x_1
(z_1, z_2) x_2

$(\underline{x_0}, \underline{x_1})$ $\underline{z_0}$
(x_1, x_2) z_1

③ Use DH transformation

$$H_1^0 = \begin{bmatrix} C_1 & -S_1 & 0 & l_1 C_1 \\ S_1 & C_1 & 0 & l_1 S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_1 = \cos\theta_1$

$S_1 = \sin\theta_1$

$\boxed{a_1, d_1, \theta_1, \alpha_1}$

$$H_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & l_1 C_1 \\ S_2 & C_2 & 0 & l_1 S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 = \cos\theta_2$

$S_2 = \sin\theta_2$

$\boxed{a_2, d_2, \theta_2, \alpha_2}$

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_1 C_1 + l_2 C_{12} \\ S_{12} & C_{12} & 0 & l_1 S_1 + l_2 S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

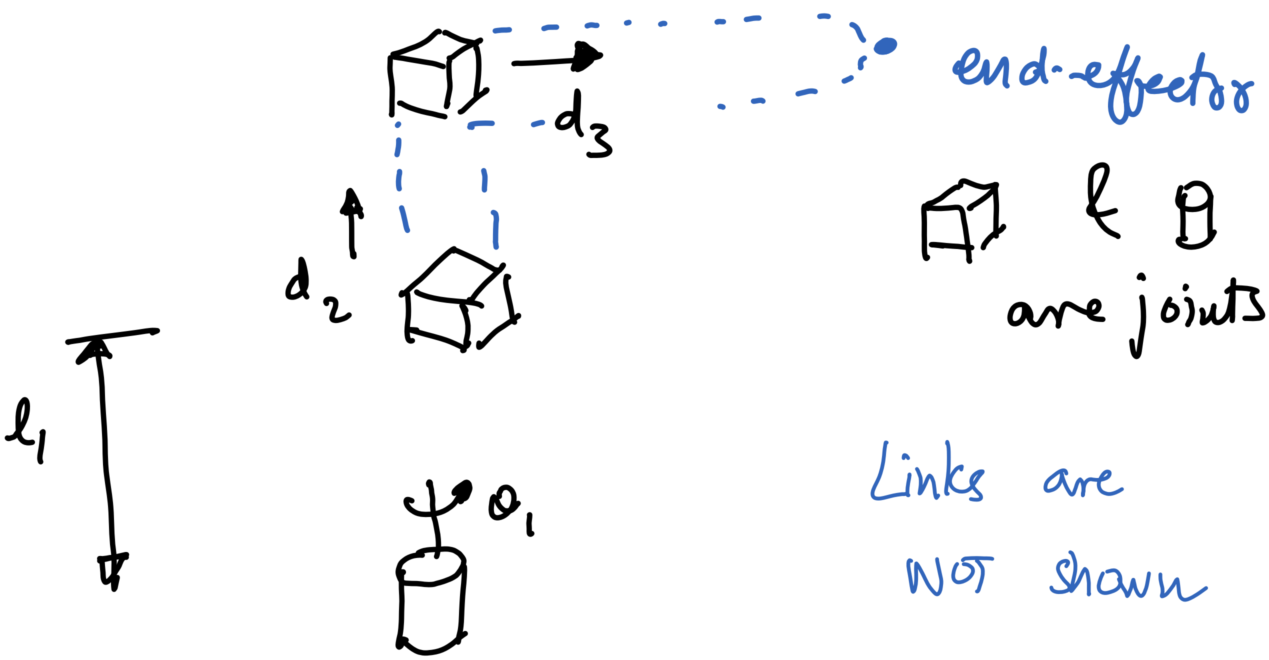
orientation of end-effector

pos of end-effector

$C_{12} = \cos(\theta_1 + \theta_2)$; $S_{12} = \sin(\theta_1 + \theta_2)$

$(\theta_1 + \theta_2)$

(2) Example 2: 3-link manipulator

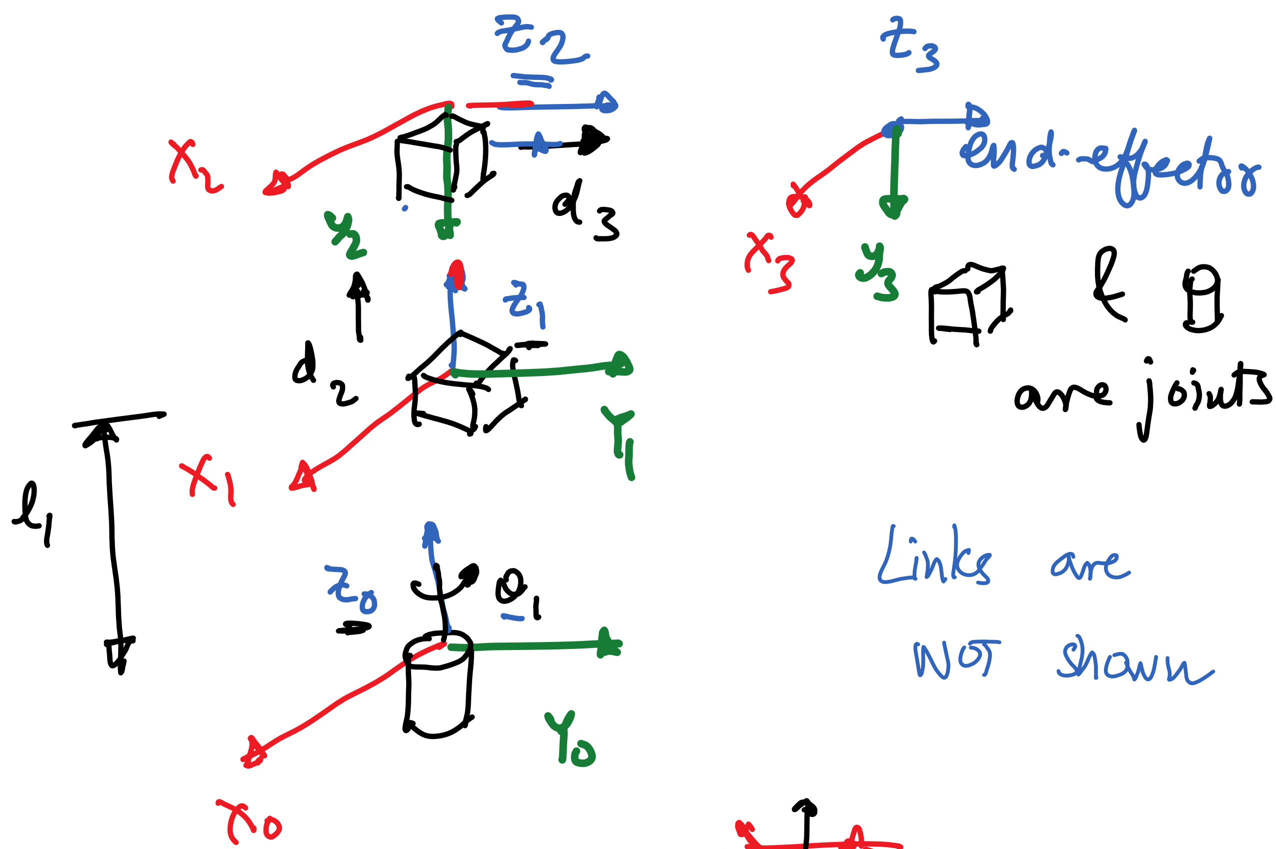


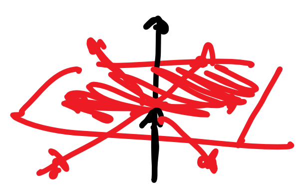
1R - 2P

1 revolute - 2 prismatic

Find the position and orientation of the end-effector

(1)





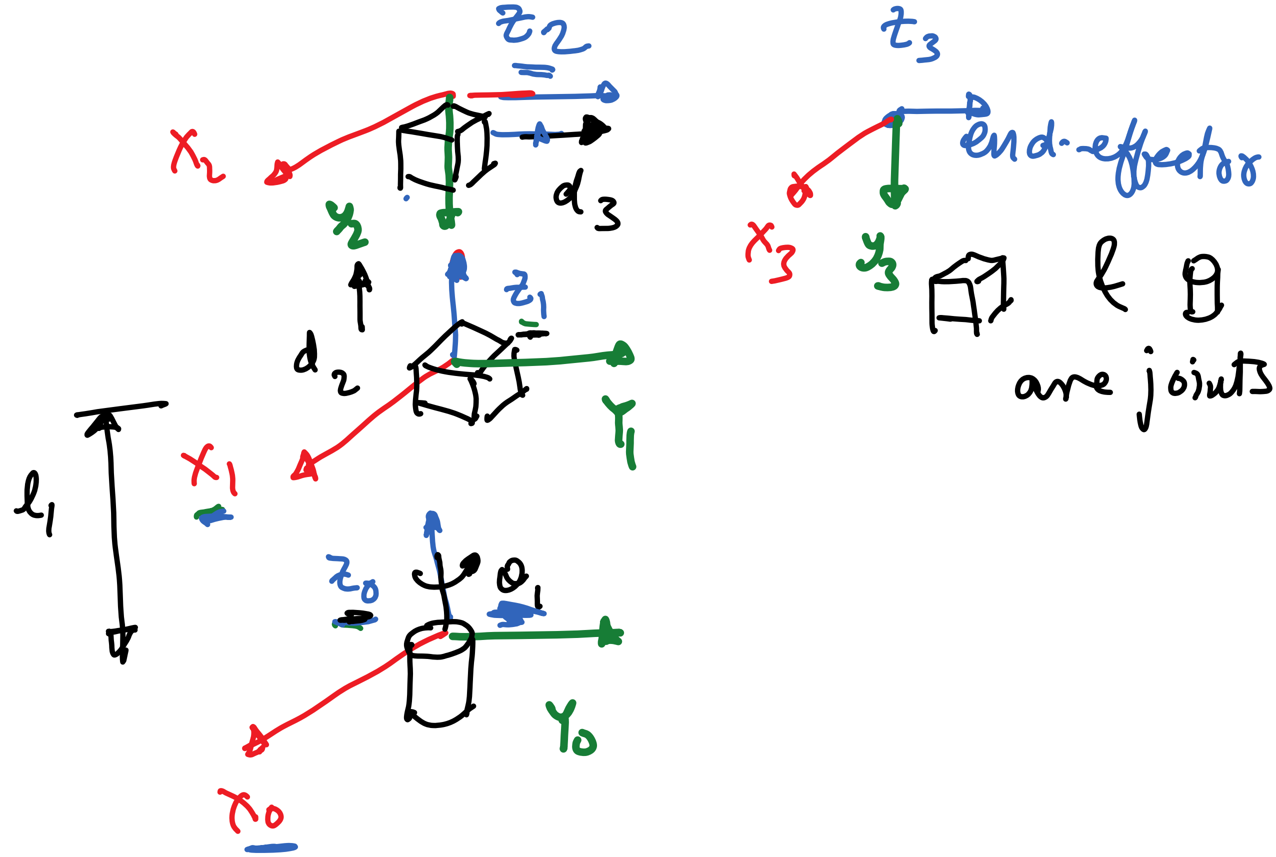
1 a) ✓

b) ✓

c) $z_0 - z_1$ intersecting

$z_1 - z_2$ intersecting

d) ✓

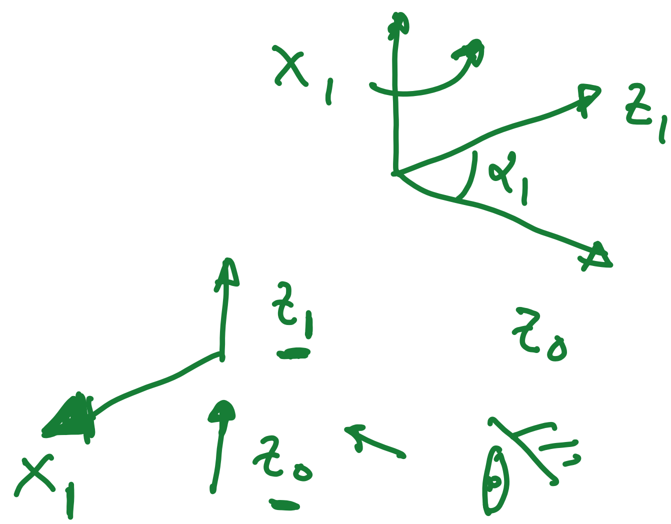


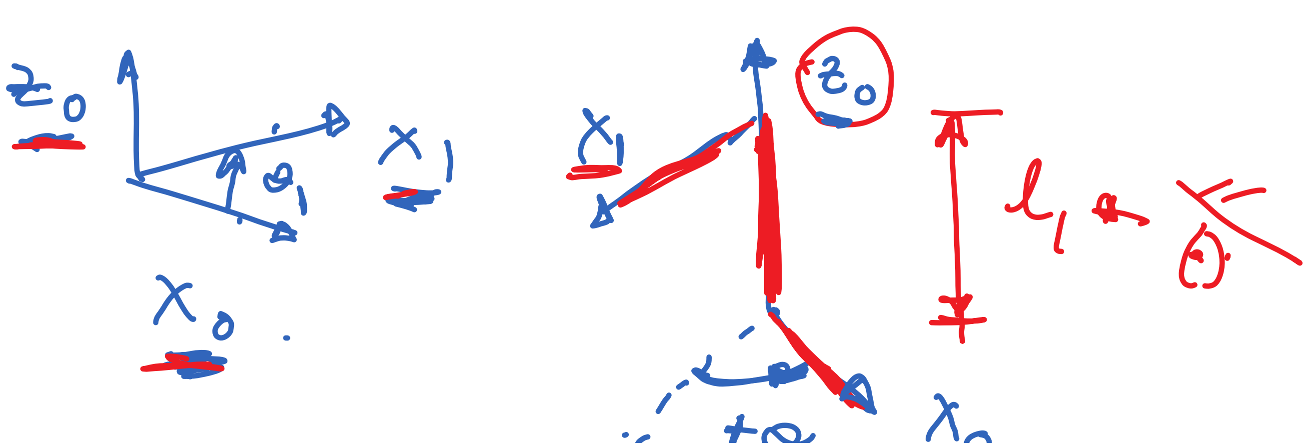
$a_1 = 0$

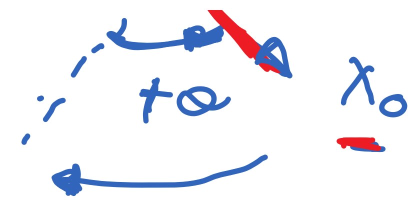
$\alpha_1 = 0$

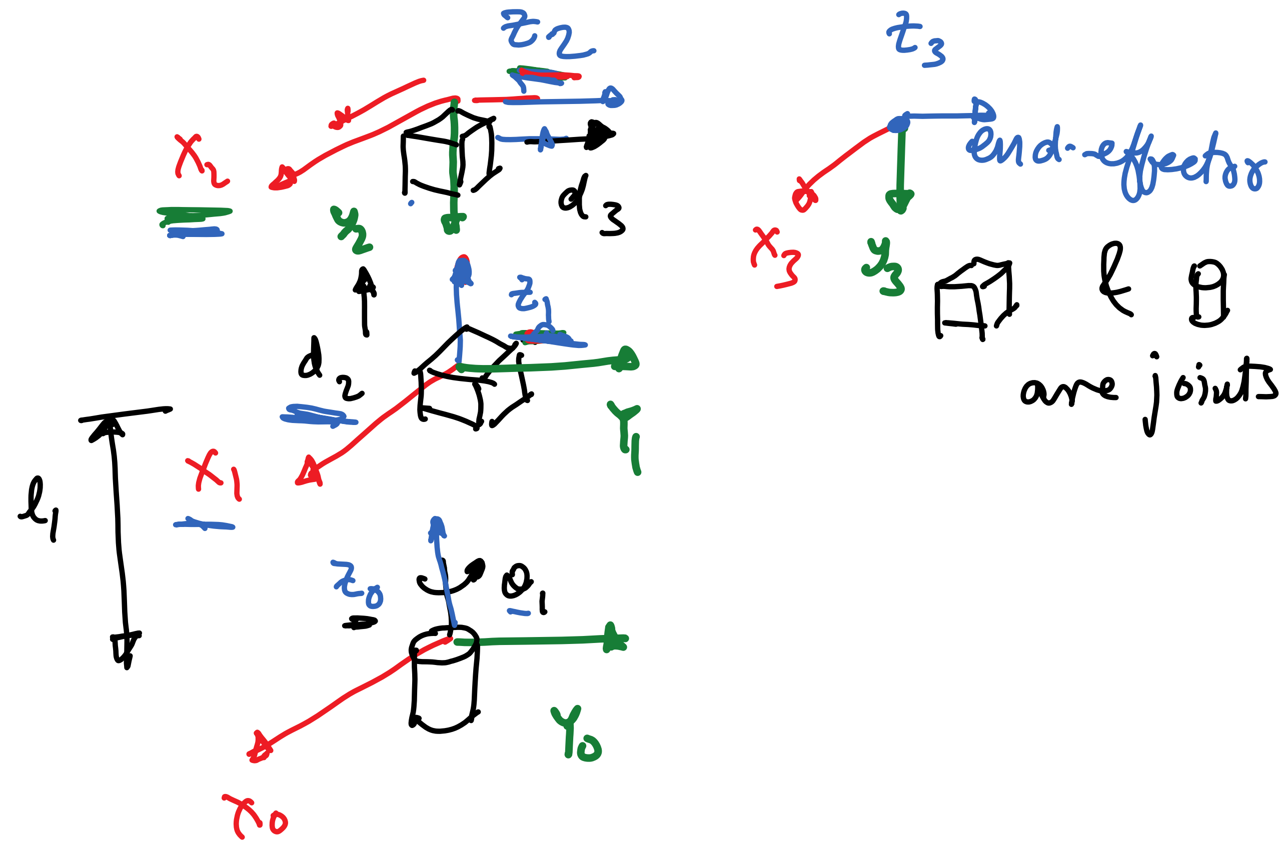
$d_1 = l_1$

$\theta_1 = \theta_1$






+θ
x₀

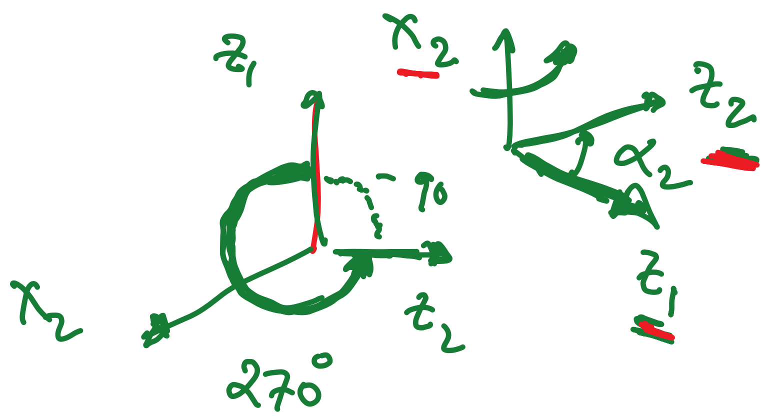


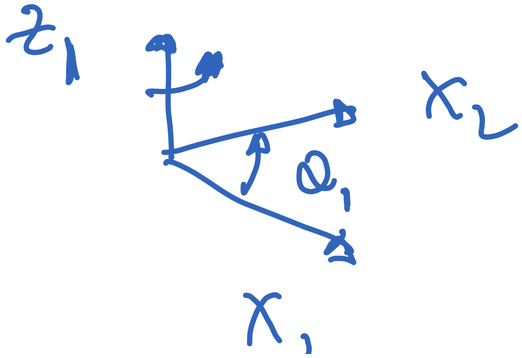
$a_2 = 0$

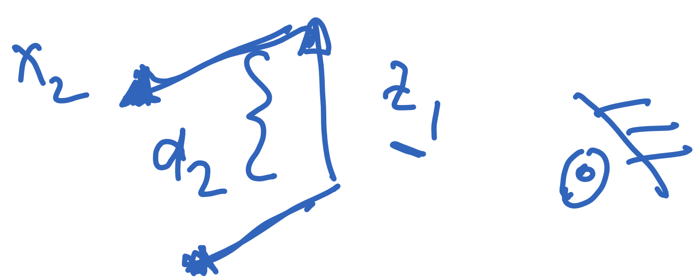
$\alpha_2 = 270^\circ$ or -90°

$d_2 = d_2$

$\theta_2 = 0$

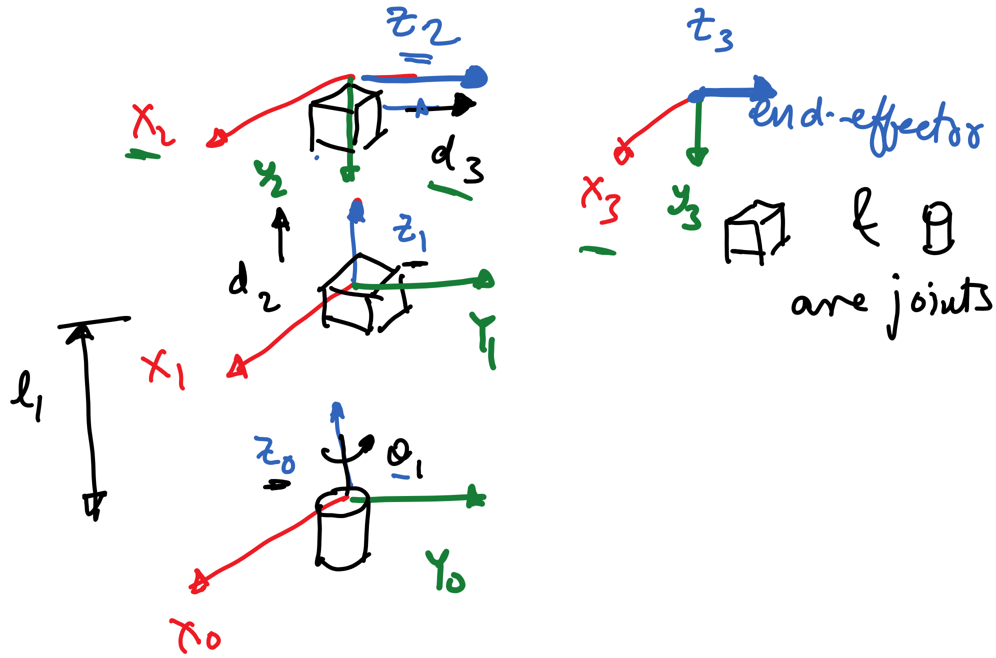






x_1

x_1

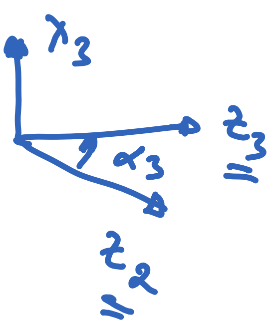


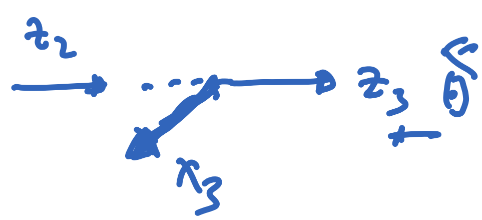
$a_3 = 0$

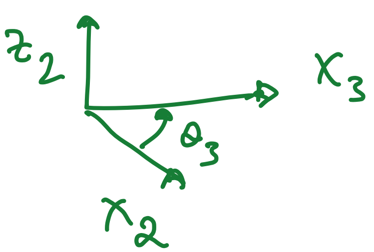
$\alpha_3 = 0$

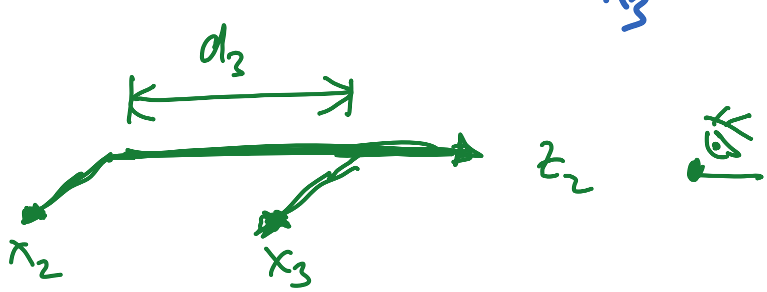
$d_3 = d_3$

$\theta_3 = 0$









(2)

Link	a_i	α_i	d_i	θ_i
1	0	0	l_1	$\sim\theta_1$
2	0	270/-90	d_2	0
3	0	0	d_3	0

$$\underline{\underline{H_i^{i-1}}} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}_{4\times 4}$$

see DH handout

$\Rightarrow H_1^0,\ H_2^1,\ H_3^2$

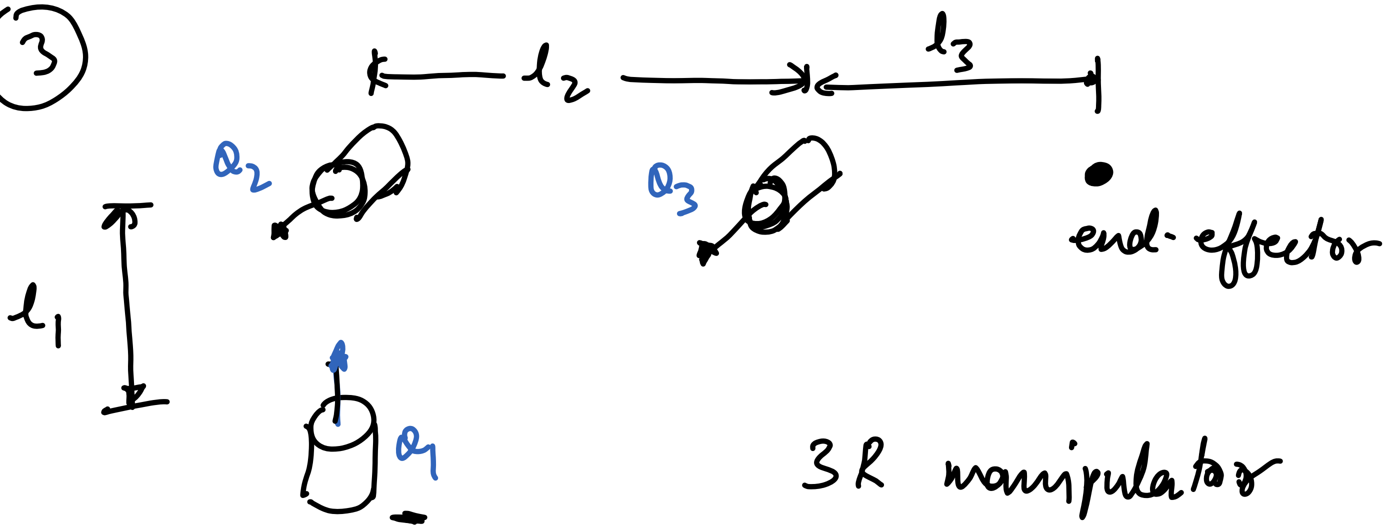
$i = 1, 2, 3$

$$H_3^0 = H_1^0 H_2^1 H_3^2 = \begin{bmatrix} (R_3^0) & (d_3^0) \\ 0 & 1 \end{bmatrix}$$

position of the end-effector

orientation of end-effector

(3)



3R manipulator
↓
revolute

Find the position and orientation of the end-effector

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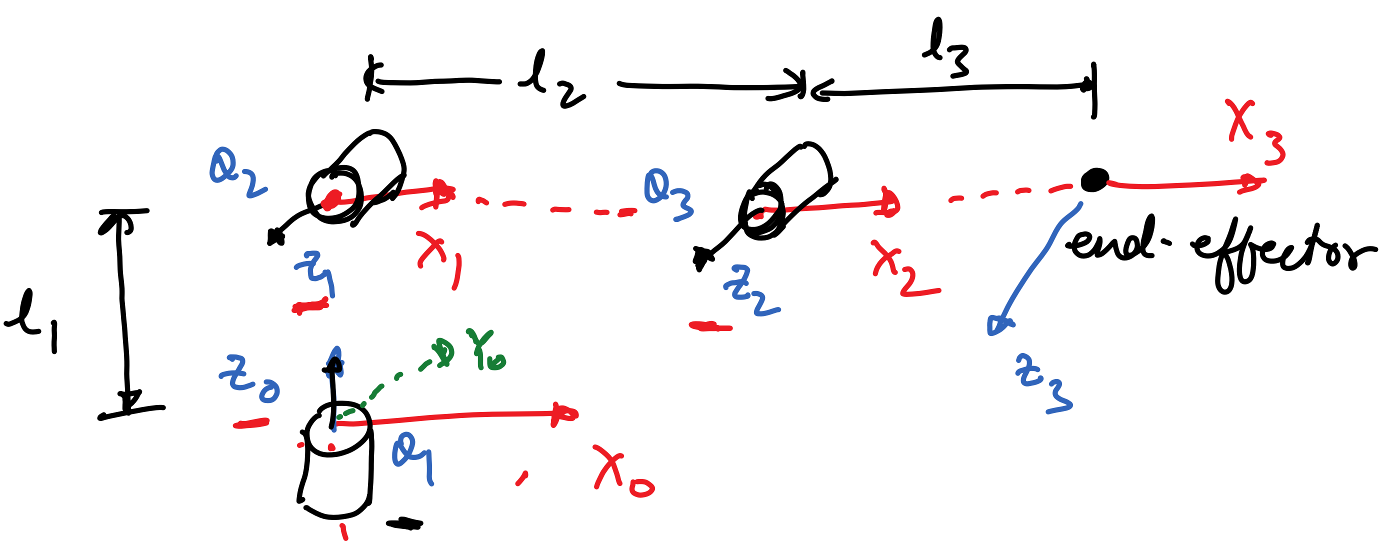
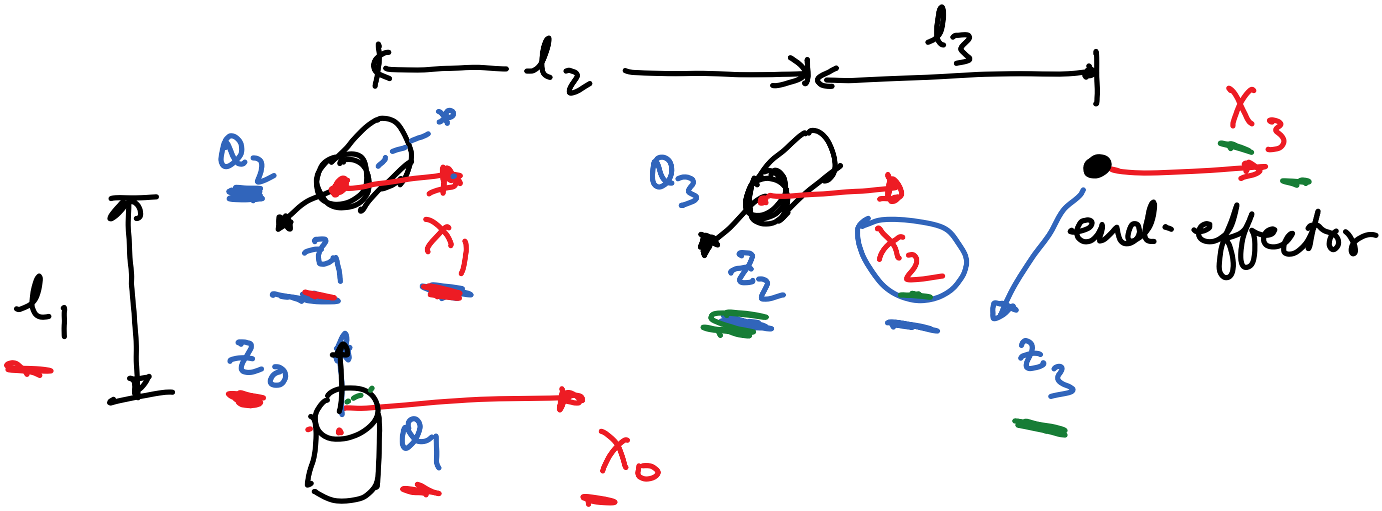
(1)

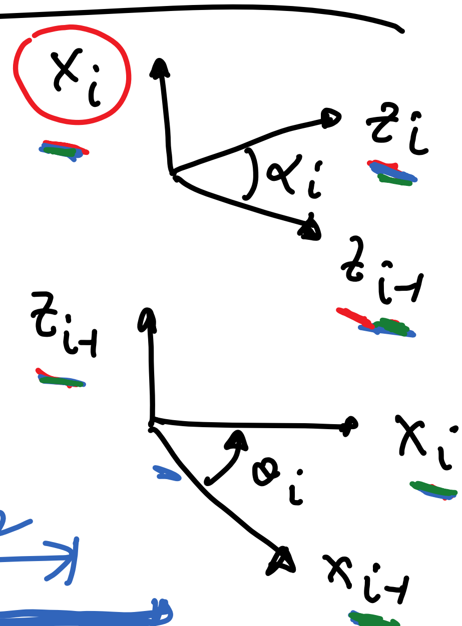


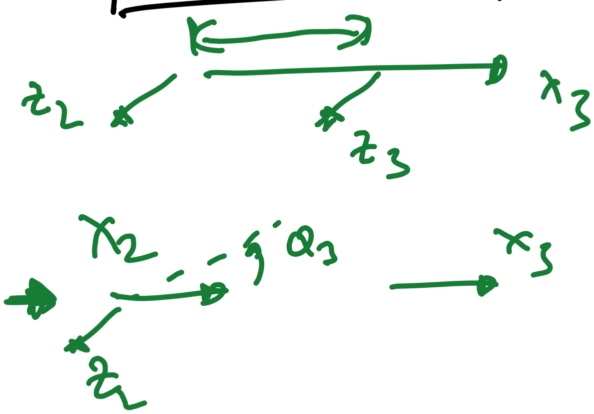
1a ✓ 1b ✓ 1c (iii) joint 0

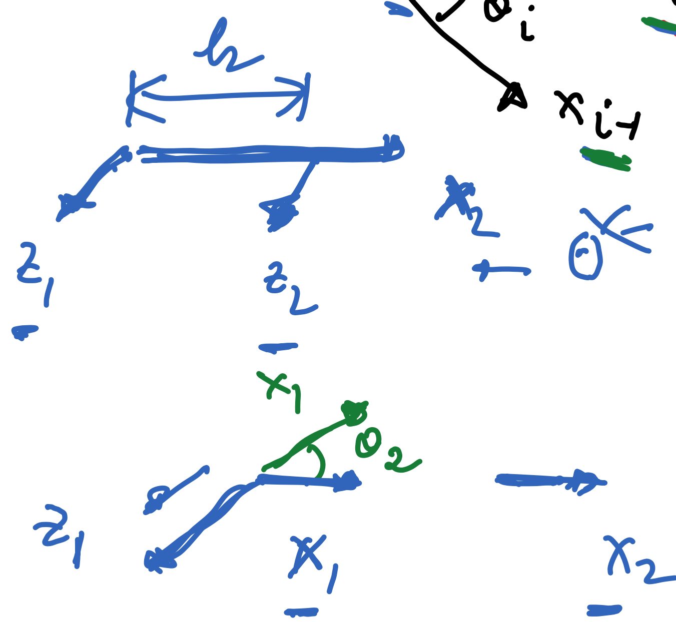
c (ii) joint 1

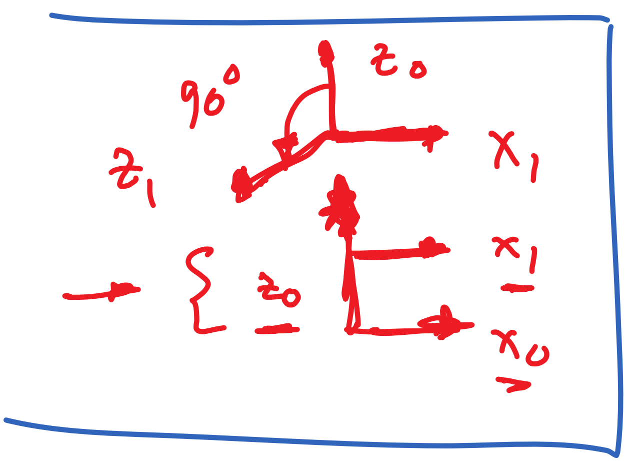


Link	a_i	α_i	d_i	θ_i
1	0	90°	l_1	θ_1
2	l_2	0	0	$-\theta_2$
3	l_3	0	0	$-\theta_3$









(3) $H_i^{i-1} = \begin{bmatrix} \; & \; & \; & \; \\ & & & \\ & & & \\ & & & \end{bmatrix}_{4\times 4}$ DH handout

$H_1^0, \; H_2^1, \; H_3^2$ ✓

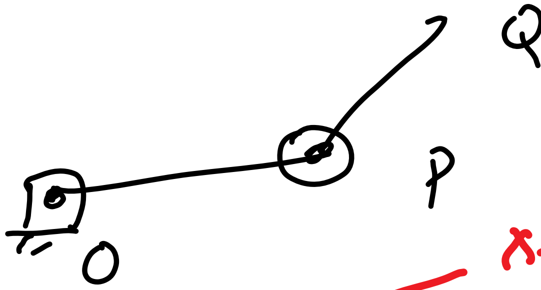
$$H_3^0 = H_1^0 \, H_2^1 \, H_3^2 = \begin{bmatrix} (R_3^0) & (d_3^0) \\ 0 & 1 \end{bmatrix}$$

position

orientation

Animation

2-D



x- coordinate   y- coordinate

line([O(1) P(1)], [O(2) P(2)])

line(P(1) Q(1)), [P(2) Q(2))

3-D

$$\text{line}(\underbrace{[O(1)\ \ P(1)]}_{x}, \underbrace{(O(2)\ \ P(2))}_{y}, \underbrace{[O(3)\ \ P(3)]}_{z})$$

$$H_1^0 = \checkmark = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix}$$ position of end-of link 1

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} R_2^0 & o_2^0 \\ 0 & 1 \end{bmatrix}$$ end-of link 2

$$H_3^0 = H_1^0 H_2^1 H_3^2 = \begin{bmatrix} R_3^0 & o_3^0 \\ 0 & 1 \end{bmatrix}$$ end of link 3

$$o_1^0 = (\underline{x}\ \ \underline{y}\ \ \underline{z})$$

and so on