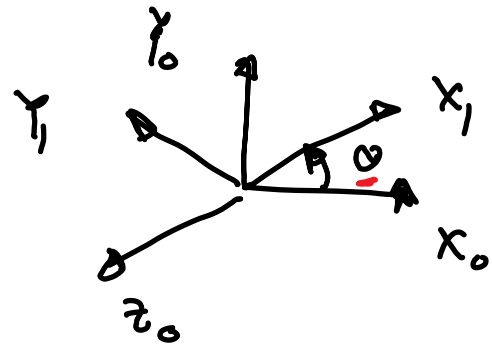
3 D angular velocity



In 2D : $\vec{\omega}_z = \dot{\theta}\,\hat{k}$

unit vector in z-direction

In 2D: $\vec{V} = \vec{\omega}_z \times \vec{r}$

same

cross product

$= J\,\dot{q}$

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$\hat{i}, \hat{j}, \hat{k}$ – unit vectors in x-, y-, z-direction

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

In 3D

$\vec{v} = \vec{\omega} \times \vec{r}$ True (Same in 2D)

$$= (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \times (r_x \hat{i} + r_y \hat{j} + r_z \hat{k})$$

cross product

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{vmatrix}$$

$$\vec{v} = \hat{i}(\omega_y r_z - \omega_z r_y)$$
$$- \hat{j}(\omega_x r_z - \omega_z r_x)$$
$$+ \hat{k}(\omega_x r_y - \omega_y r_x)$$

Skew symmetric matrix

$$S(a) + S^T(a) = 0$$

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Property

① $\vec{a} \times \vec{b} = S(a)\, b$ — we will prove this.

② $R\, S(a)\, R^T = S(Ra)$

R = rotation matrix

$$v = \vec{\omega} \times \vec{r} = S(\omega)\, r = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$= \begin{bmatrix} -\omega_z r_y + \omega_y r_z \\ \omega_z r_x - \omega_x r_z \\ \cdots \end{bmatrix}$$ ✓ checks with $\vec{\omega} \times \vec{r}$

$$\begin{bmatrix} \omega_z r_x - \omega_x r_z \\ -\omega_y r_x + \omega_x r_y \end{bmatrix}$$

with

$\vec{\omega} \times \vec{r}$

3D: $\vec{\omega} \neq \dot{\phi}\,\hat{i} + \dot{\theta}\,\hat{j} + \dot{\psi}\,\hat{k}$

rate of change of euler angles

Because:
rotations are not commutative

Angular velocities for 3-2-1 euler angles

$$R R^T = I$$

Diff. w.r.t. time $\quad \dot{R} R^T + R \dot{R}^T = 0$

$$\dot{R} R^T + \left(\boxed{\left(R \dot{R}^T\right)^T}\right)^T = 0$$

$(AB)^T = B^T A^T$

$$\dot{R} R^T + \left(\dot{R} R^T\right)^T = 0$$

$$S(a) + S(a)^T = 0$$

skew symmetric matrix.

$$S(a) = \dot{R} R^T$$

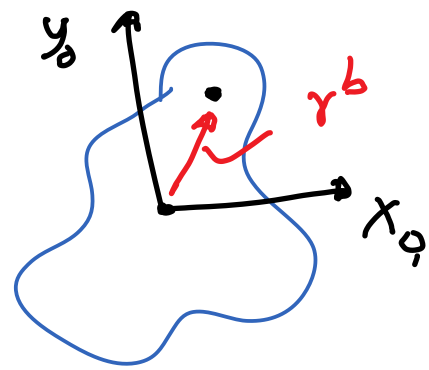
$$\dot{R} R^T = S(a)$$

Post. multiply with R

$$\dot{R} \underbrace{R^T R} = S(a) R$$

$$\boxed{\dot{R} = S(a) R} \quad \text{— (I)}$$

Establish what a ?



$$r = \underline{R r_b} \quad \text{— (II)}$$

↑
position
in world frame

Diff. w.r.t. time

$$\dot{r} = \dot{R} r_b + R \cancelto{0}{\dot{r}_b}$$

From (I)

$$\dot{r} = S(a) R r_b$$

From (II) $\dot{r} = S(a) r$

$$\dot{r} = S(a)\, r$$

$$= \vec{a} \times \vec{r} \quad \text{— III} \qquad S(a)b = \vec{a} \times \vec{b}$$

We know that $\vec{v} = \vec{\omega} \times \vec{r}$ IV

From III and IV $\vec{\omega} = \vec{a}$

$$\boxed{\dot{R} R^T = S(\omega) \ ; \quad \dot{R} = S(\omega) R}$$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

How is $\omega_x, \omega_y, \omega_z$ related to $\dot{\phi}, \dot{\theta}, \dot{\psi}$ (rate of change of Euler angles)?

$$S(\omega) = \dot{R} R^T \qquad \{R = R_z R_y R_x\}$$

3-2-1

$$= \dot{\overline{(R_z R_y R_x)}} \, (R_z R_y R_x)^T$$

$$= \dot{\overline{(R_z R_y R_x)}} \, R_x^T R_y^T R_z^T$$

$$= \left(\underbrace{\dot{R}_z R_y R_x}_{1} + \underbrace{R_z \dot{R}_y R_x}_{2} + \underbrace{R_z R_y \dot{R}_x}_{3}\right) \underline{\underline{R_x^T R_y^T R_z^T}}$$

$$① \quad \dot{R}_z \underbrace{R_y \underbrace{R_x R_x^T}_{I} R_y^T}_{I} R_z^T = \dot{R}_z R_z^T$$

$$= S(\omega_z) = S(\dot{\psi} \hat{k})$$

As $\dot{R} R^T = S(a)$

(2) $R_z \dot{R}_y \underbrace{R_x R_x^T}_{I} R_y^T R_z^T = R_z \dot{R}_y R_y^T R_z^T$

$\dot{R}_y R_y^T = S(\omega_y) = S(\dot{\theta}\hat{j})$

$R_z \, S(\dot{\theta}\hat{j}) \, R_z^T$

But $\quad \underline{R} \; \underline{S(a)} \; \underline{R^T} = S(Ra)$

$S(R_z \dot{\theta}\hat{j}) \quad \oplus$

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(3) $R_z R_y \underbrace{\dot{R}_x R_x^T}_{S(\omega_x) \; = \; S(\dot{\phi}\hat{i})} R_y^T R_z^T$

$R_z R_y \; S(\dot{\phi}\hat{i}) \; R_y^T \; R_z^T = R_z R_y \, S(\dot{\phi}\hat{i}) (R_z R_y)^T$

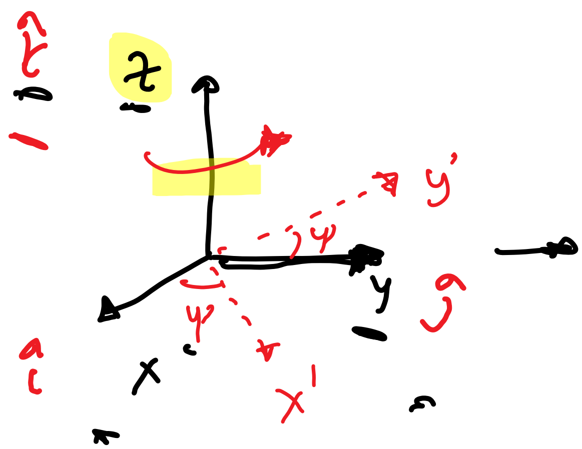
$S(R_z R_y \dot{\phi}\hat{i}) \quad (*)$

$$S(\omega) = S(\dot{\psi}\hat{k}) + S(R_z\dot{\theta}\hat{j}) + S(R_zR_y\dot{\phi}\hat{i})$$

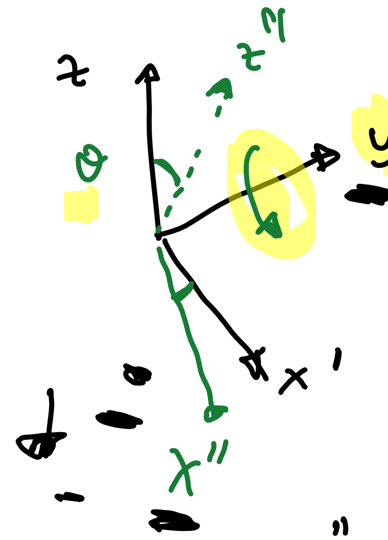
$$S(\omega) = S(\dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_zR_y\dot{\phi}\hat{i})$$

$$\boxed{\vec{\omega} = \dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_zR_y\dot{\phi}\hat{i}}$$

Angular velocity (another way)

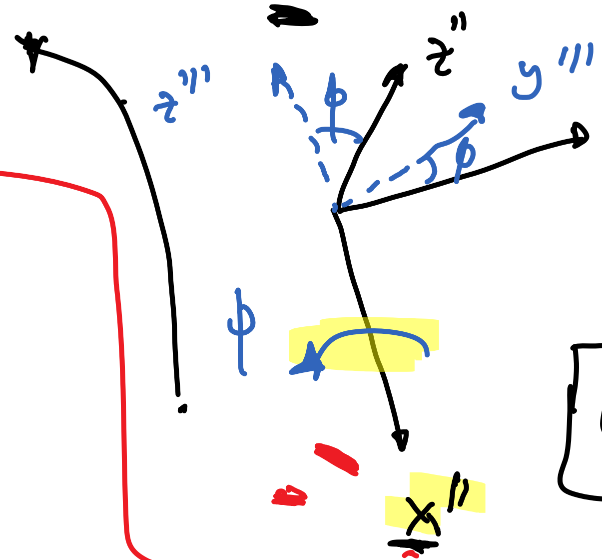


$\boxed{C^0 \to C^1}$



$\boxed{C_1 \to C_2}$

$$y' = R_z y$$



$\boxed{C_2 \to C_3}$

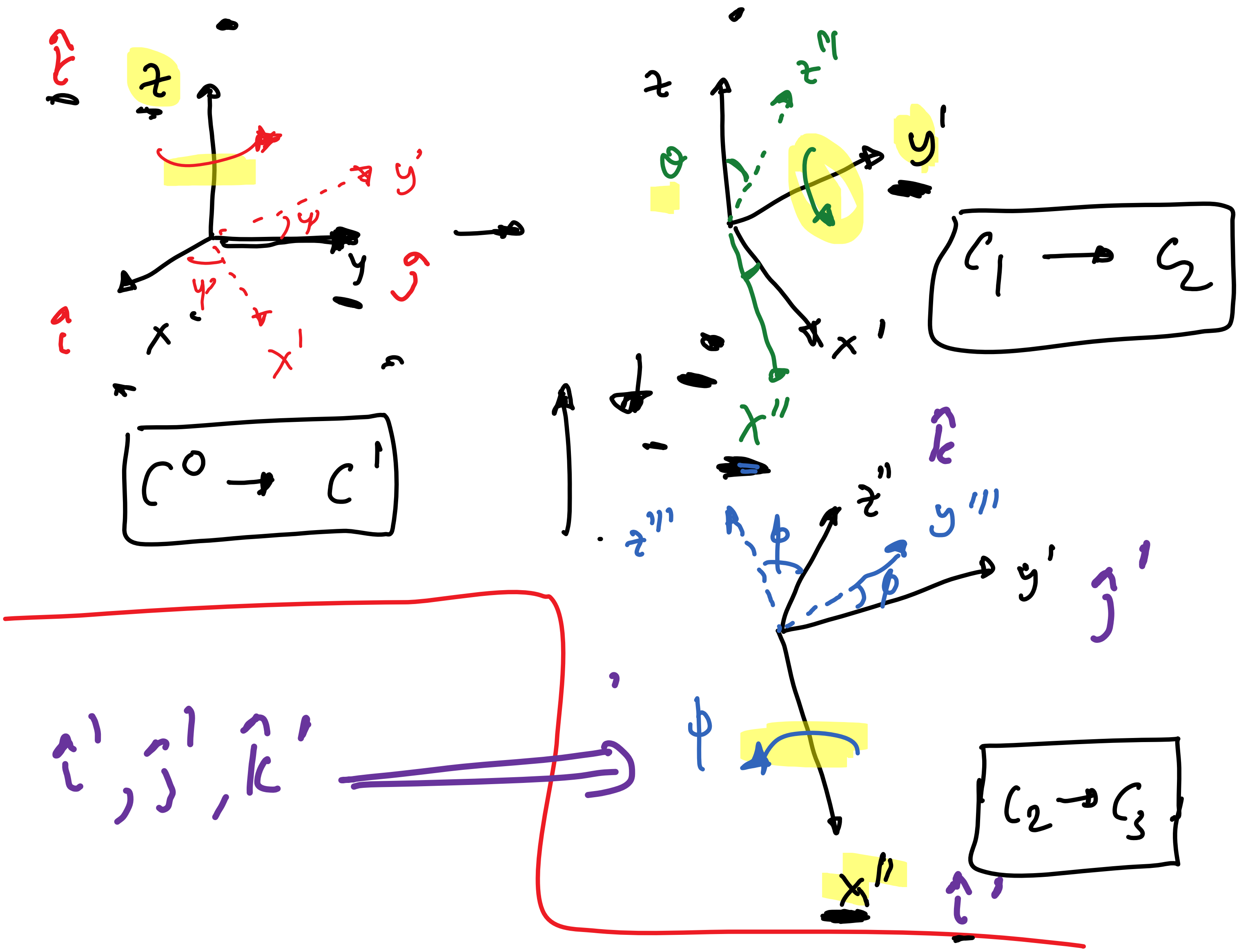
$$\vec{\omega} = \dot{\psi}\, z + \dot{\theta}\, y' + \dot{\phi}\, x''$$

$$= \dot{\psi}\, \hat{k} + \dot{\theta}\, (R_z \hat{j}) + \dot{\phi}\, (R_z R_y \hat{i})$$

$$\boxed{\vec{w} = \dot{\psi}\, \hat{k} + \dot{\theta}\, R_z \hat{j} + \dot{\phi}\, R_z R_y \hat{i}}$$

same as before

world frame angular velocity



$$\vec{w}_b = \dot{\phi}\, x'' + \dot{\theta}\, y' + \dot{\psi}\, z$$

$$= \dot{\phi}\, \hat{i}' + \dot{\theta}\, R_x^T \hat{j}' + \dot{\psi} R_x^T R_y^T \hat{k}'$$

$$\boxed{\vec{w}_b = \dot{\phi}\, \hat{i}' + \dot{\theta}\, R_x^T \hat{j}' + \dot{\psi} R_x^T R_y^T \hat{k}'}$$

$$\omega = \begin{bmatrix} \cos\psi \cos\theta & -\sin\psi & 0 \\ \cos\theta \sin\psi & \cos\psi & 0 \\ -\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\omega_b = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\phi \cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

$$\omega = A\dot\Theta$$

$$\omega_b = A_b\dot\Theta$$

rate of change of euler angles

$$\dot\Theta = A^{-1}\omega$$

$$\dot\Theta = A_b^{-1}\omega_b$$

is singular when $\cos\theta = 0$ or $\theta = \pi/2$

Use quarternions to avoid

singularities.