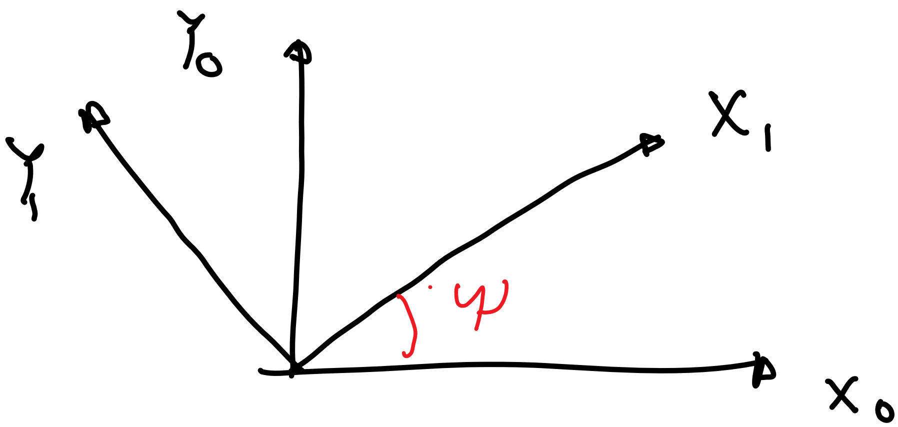
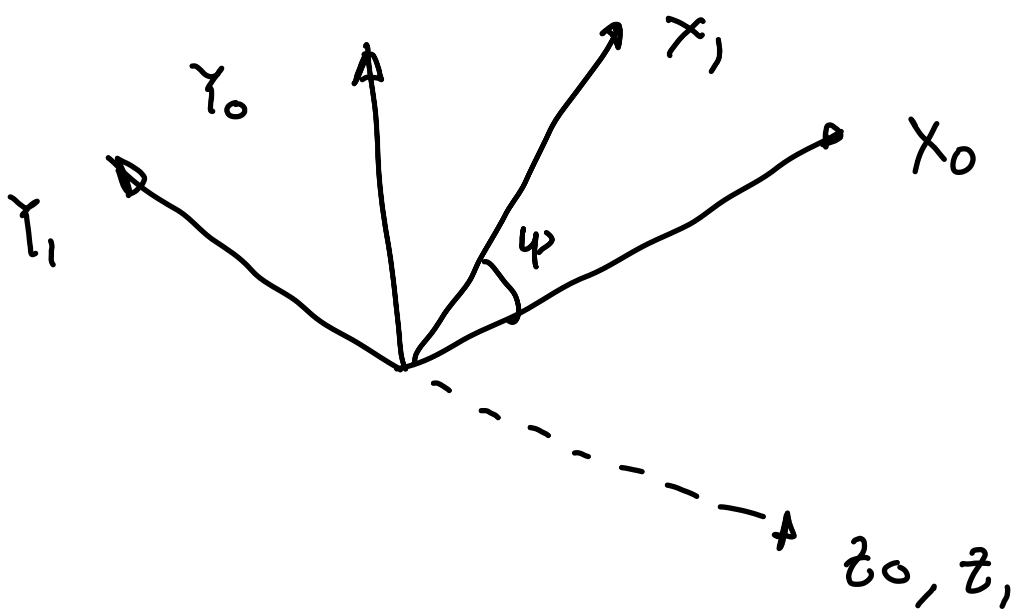
3 D rotations



$$R_1^0 = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$



$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

3D rotation

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

not an error

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

In general rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \text{9 numbers}$$

$$R^T R = I = R R^T \quad (I = 3\times3 \text{ identity matrix})$$

$$3\left\{\begin{array}{l} r_{11}^2 + r_{21}^2 + r_{31}^2 = 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 = 1 \\ r_{13}^2 + r_{23}^2 + r_{33}^2 = 1 \end{array}\right\}$$

Rotation matrix has unit magnitude

$$3\left\{\begin{array}{l} \sum_{i=1,2,3} r_{i1} r_{i2} = 0 \quad = r_{11} r_{12} + r_{21} r_{22} + r_{31} r_{32} = 0 \\ \sum_{i=1,2,3} r_{i2} \, r_{i3} = 0 \quad \Rightarrow \\ \sum_{i=1,2,3} r_{i3} \, r_{i1} = 0 \quad \Rightarrow \end{array}\right\}$$

Orthogonality of the rotation matrix

9 constants − 6 conditions = 3 unique numbers

Euler angles to parameterize rotations.

Euler angles – 3 angles

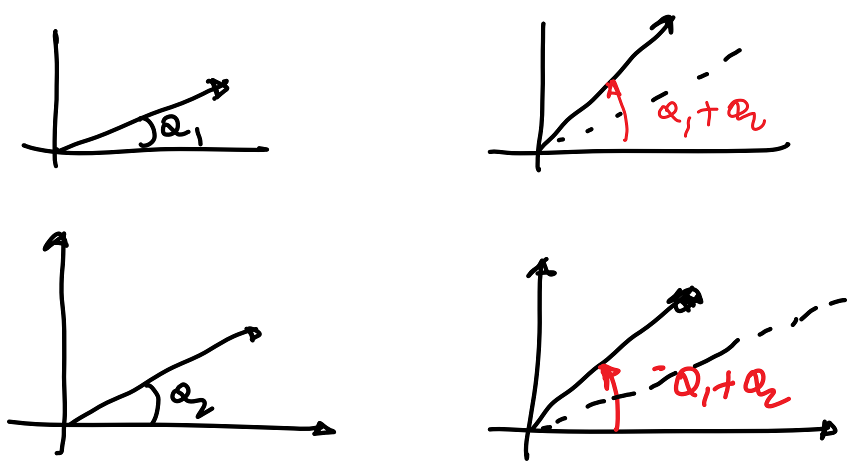
Tait-Bryan / aerospace

1-2-3 Bryant angles

$\boxed{x-y-z}$	$y-x-z$	$\boxed{z-x-y}$
$x-z-y$	$y-z-y$	$\boxed{z-y-x}$
$x-y-x$	$y-z-x$	$z-y-z$
$x-z-x$	$y-x-y$	$z-x-z$
4 +	4 +	4 = 12

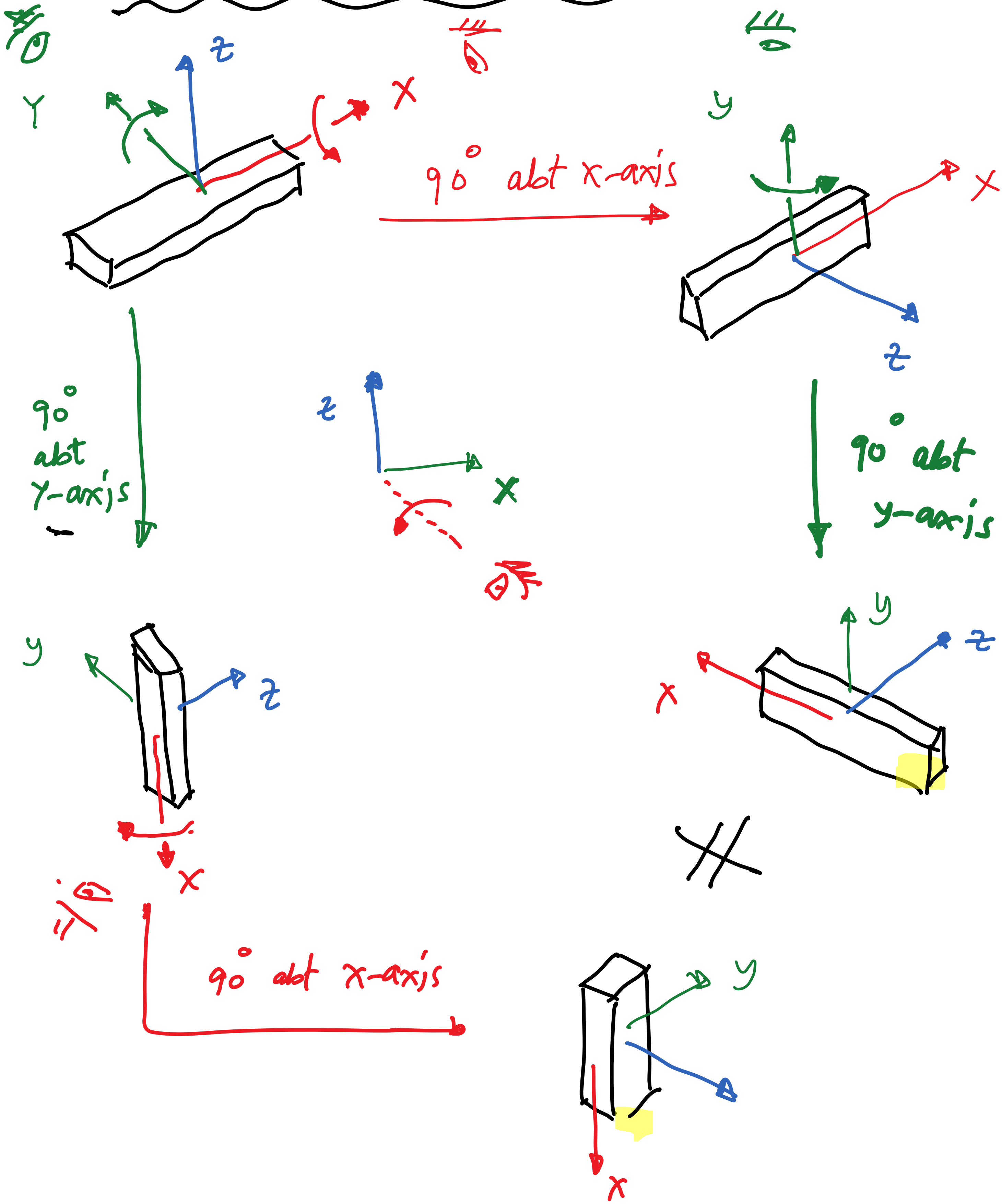
we will use this one (3-2-1)

Rotations in 2D are commutative



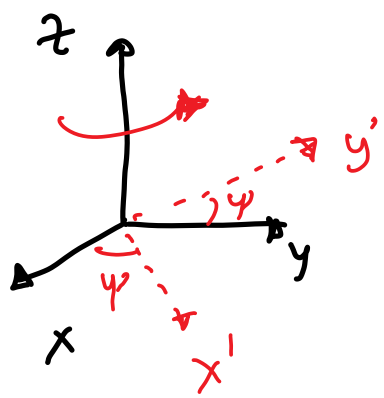
$R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$ $\qquad$ $R_{\theta_2} R_{\theta_1} = R_{\theta_1+\theta_2}$

Rotations in 3D are not commutative

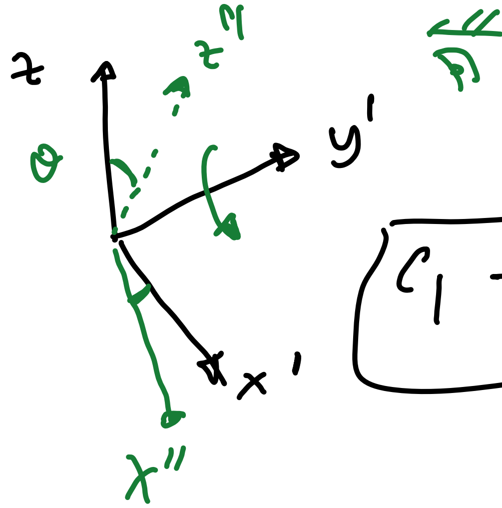


3-2-1 Euler-angles (z-y-x)

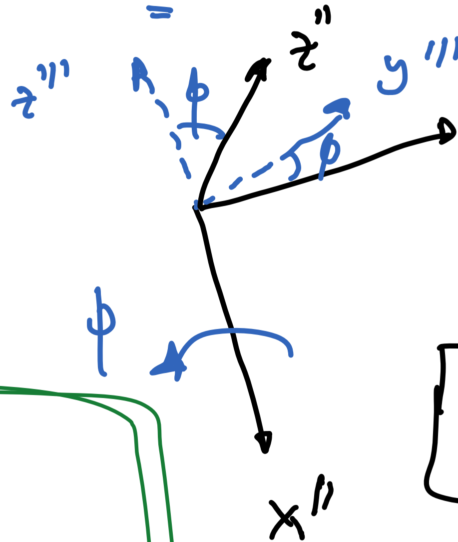
$\psi - \theta - \phi$



$\boxed{C^0 \rightarrow C^1}$



$\boxed{C_1 \rightarrow C_2}$



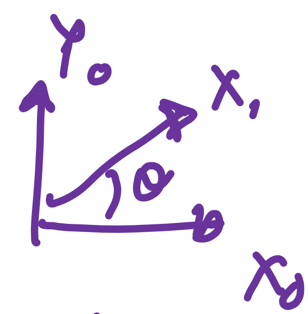
$\boxed{C_2 \rightarrow C_3}$

$$\Rightarrow C^0 = R_z(\psi)\, C^1$$

$$C^1 = R_y(\theta)\, C^2$$

$$C^2 = R_x(\phi)\, C^3$$

$$C^0 = R_z R_y R_x\, C^3$$



$$C^0 = R_1^0\, C^1$$

2-D

2-D

$$C^0 = R_z(\psi)\, R_y(\theta)\, R_x(\phi)\, C^3$$

$$C^0 = R \; C^3$$

fixed frame/
world frame

rotation matrix

body frame

$$\boxed{r^0 = R\, r^{body}}$$

r^0 — position in world frame

r^{body} — position in body frame

R — rotation matrix

$$R = R_z(\psi)\, R_y(\theta)\, R_x(\phi$$

$$= \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\psi\cos\phi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$