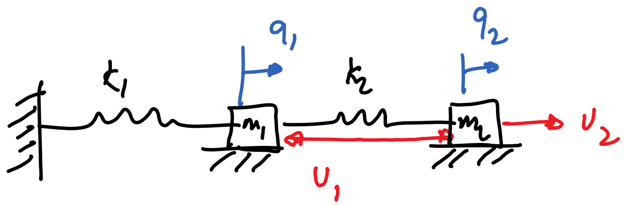
State Estimation



$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -k_2/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/m_1 & 0 \\ 1/m_2 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$x_1 = q_1$; $x_2 = q_2$; $x_3 = \dot{q}_1$; $x_4 = \dot{q}_2$

$u = -Kx$ { Pole placement / LQR to compute K }

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\underset{2\times 4}{K} \, [x_1, x_2 \; x_3 \; x_4]_{4\times 1}$$

We need all 4 states for control.
This means we need 4 sensors (2 - position, 2- velocity).

Since we position is related to velocity, we can get away with only 2 sensors and estimate the remaining 2. This will make the design cheaper.

y - sensor measurements

We will assume we have velocity sensors installed.

$$y = Cx$$

$$\underset{2\times1}{y} = \underset{2\times4}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

We want to estimate $\hat{x}_1$, $\hat{x}_2$ from measurements of $\hat{x}_3$, $\hat{x}_4$

Estimated state $\hat{x} = (\hat{x}_1 \;\; \hat{x}_2 \;\; \hat{x}_3 \;\; \hat{x}_4)$

$$\dot{x} = Ax + Bu \quad \text{(True)}$$

$$\dot{\hat{x}} = A\hat{x} + Bu \quad \text{(Estimate)}$$

$$e = x - \hat{x}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu}$$

$$= A(x - \hat{x})$$

$$= Ae$$

$$\dot{e} = Ae$$

This error depends on A (system dynamics)

If A is unstable e will also be unstable.

Luenberger Observer (Deterministic)

Consider the following estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

error in measured state & estimate state

user-defined gain

$$\dot{\hat{x}} = A\hat{x} + Bu + L(Cx - C\hat{x})$$

$$\dot{\hat{x}} = LCx + (A-LC)\hat{x} + Bu$$

$$e = x - \hat{x}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - \{LCx + (A-LC)\hat{x} + \cancel{Bu}\}$$

$$= (A-LC)x - (A-LC)\hat{x}$$

$$\dot{e} = (A-LC)(x-\hat{x})$$

$$\Rightarrow \dot{e} = (A-LC)e$$

$$\dot{x} = (A-BK)x$$

We will choose L to place the eigenvalues of $A-LC$ far away on the -ive plane

Observability

A linear system is observable if and only if any initial state $x(0)$ can be reconstructed from the output $y(t)$ within a finite time $(t - 0)$

$$Ob = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

observability matrix

rank $(Ob) = n$ then the system is observable

$< n$ then the system is NOT observable

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import control
setup A, C
Ob = control.obsv(A, C)
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```
np.linalg.matrix_rank(Ob)
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Controllability / Observability Dual

1 – $C_o = [B, AB, \ldots A B^{n-1}]$

$$Ob = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$Ob^T = [C^T \; (CA)^T \; (CA^2)^T \ldots \; (CA^{n-1})^T]$$

2 – $Ob^T = [C^T \;\; A^T C^T \;\; (A^2)^T C^T \ldots (A^{n-1})^T C^T]$

$C_o \simeq Ob^T$

$B = C^T$

$A = A^T$

$\uparrow \qquad \uparrow$

$C_o \qquad Ob$

$\to Ob^T = \text{control.ctrb}(A^T, C^T)$

$\to C_o = \text{control.ctrb}(A, B)$

③ — $\dot{x} = (\underline{A} - \underline{B}K)x \Rightarrow K = place(A, B, p)$

User chosen

$\dot{e} = (A - LC)e$

$\dot{e}^T = (A - LC)^T e^T$

④ — $\dot{e}^T = (A^T - C^T L^T) e^T$

$$\left.\begin{aligned} x &\longleftrightarrow e \\ A &\longleftrightarrow A^T \\ B &\longleftrightarrow C^T \\ K &\longleftrightarrow L^T \end{aligned}\right\} L^T = place(A^T, C^T, p)$$

② Kalman Filter (Linear Quadratic Estimator)

[Stochastic system]

$$\dot{x} = Ax + Bu + Gw$$ process noise

$$y = Cx + Du + \underline{v}$$ sensor noise

$$E(ww^T) = Q_e$$
$$E(vv^T) = R_e$$
$$E(wv^T) = N_e$$

} stochastic correlation

Estimate $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$= A\hat{x} + Bu + L(y - C\hat{x} - Du)$$

Error dynamics

$$e = x - \hat{x}$$

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$

$$\dot{e} = (Ax + Bu + Gw) - [A\hat{x} + Bu + L(\underbrace{y}_{Cx+Du+v} - C\hat{x} - Du)]$$

$$\dot{e} = Ax + \cancel{Bu} + Gw - A\hat{x} - \cancel{Bu} \ldots$$
$$- L(Cx + \cancel{Du} + v - C\hat{x} - \cancel{Du})$$

$$\dot{e} = (A - LC)e + Gw - Lv$$

affected

process noise

sensor noise

To compute L we min $J = \lim_{T \to 0} \frac{1}{T} \int_0^T e^T e$

$$AP + PA^T - (PC^T + GN_e) R_e^{-1} (CP + N^T G) + GQ_e G^T = 0$$

Ricatti equation

✓

$L, P, E =$ control.lqe$(A, G, C, \underbrace{Q_e, R_e, N_e}_{\text{stochasticity}})$

Stochasticity

Linear Quadratic Gaussian (LQG)

control

estimation

LQR

LQE / Kalman filter

$$u = -K\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

estimate