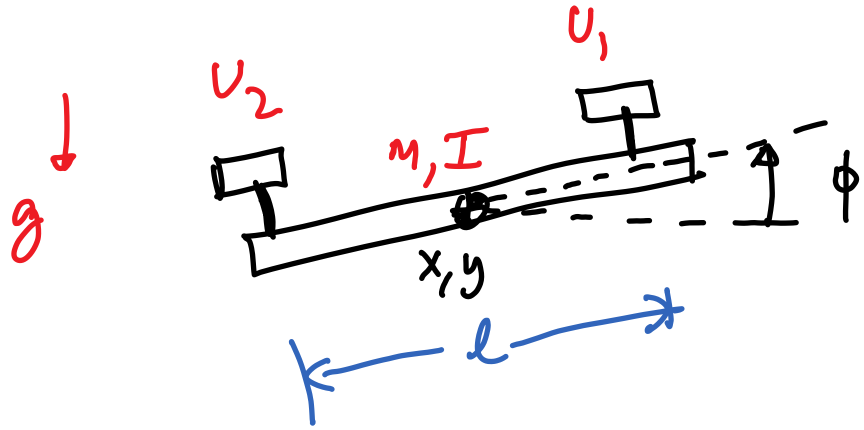
Bicopter



m, I – mass, inertia

g, l – gravity, length

U_1, U_2 – thrust forces

x, y, ϕ – degrees of freedom

Euler–Lagrange

1. Get positions / velocities

x, y, θ

$\dot{x}, \dot{y}, \dot{\theta}$

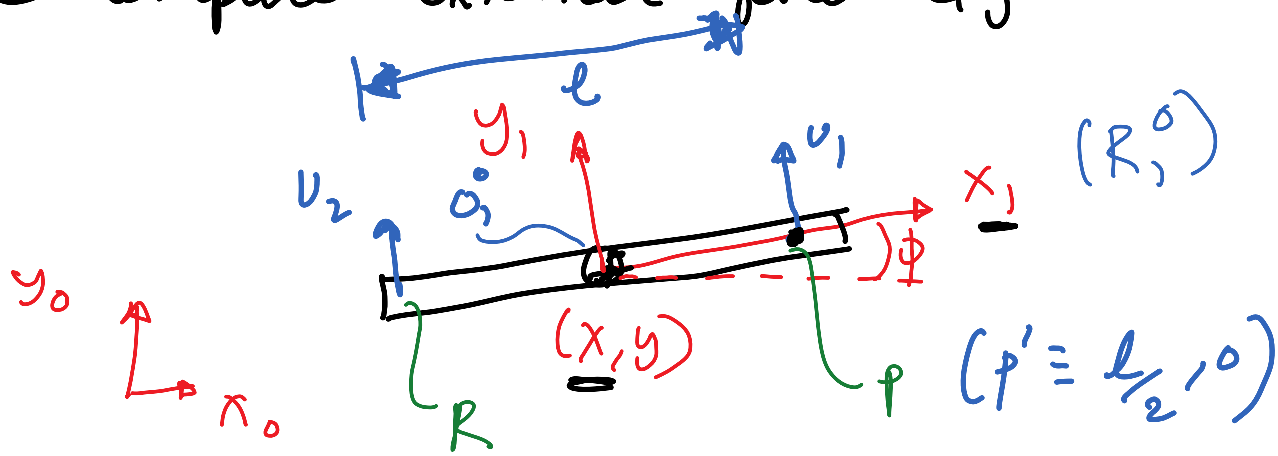
② Get kinetic and potential energy

$$T = 0.5\, m\,(\dot{x}^2 + \dot{y}^2) + 0.5\, I\, \dot{\phi}^2$$

$$V = m g y$$

$$\mathcal{L} = T - V$$

③ Compute external forces Q_j



$$Q_j = J_p^T F_p^0 + J_R^T F_R^0$$

frame 0

point R

jacobian of point R

$$H_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$p^0 = H_1^0 \, p^1 = H_1^0 \begin{bmatrix} l/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x + 0.5\,l\cos\phi \\ y + 0.5\,l\sin\phi \\ 1 \end{bmatrix}$$

$$r^0 = H_1^0 \, r^1 = H_1^0 \begin{bmatrix} -l/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x - 0.5\,l\cos\phi \\ y - 0.5\,l\sin\phi \\ 1 \end{bmatrix}$$

$$J_p = \frac{\partial p^0}{\partial X} = \begin{bmatrix} 1 & 0 & -0.5\,l\sin\phi \\ 0 & 1 & 0.5\,l\cos\phi \end{bmatrix}$$

$X \rightarrow \{x, y, \theta\}$

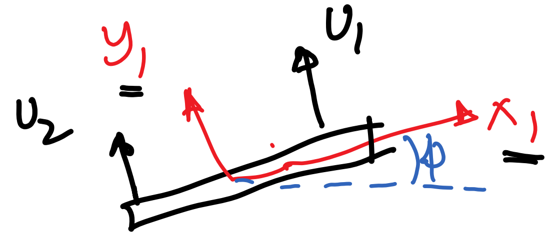
2×3

$\begin{pmatrix} x + 0.5\,l\cos\phi \\ y + 0.5\,l\sin\phi \end{pmatrix}$ $\quad (x, y, \theta)$

$$J_R = \frac{\partial r^0}{\partial X} = \begin{bmatrix} 1 & 0 & 0.5\,l\sin\phi \\ 0 & 1 & -0.5\,l\cos\phi \end{bmatrix}$$

2×3

$$F_p^0 = R_1^0 \, F_p^1$$



$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ U_1 \end{bmatrix} = \begin{bmatrix} -U_1 \sin\phi \\ U_1 \cos\phi \end{bmatrix}$$

$$F_R^0 = R_1^0 \, F_R^1$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ U_2 \end{bmatrix} = \begin{bmatrix} -U_2 \sin\phi \\ U_2 \cos\phi \end{bmatrix}$$

$$Q_j = \underline{\underline{J_p^T}} \, F_p^0 + J_R^T \, F_R^0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{l}{2}\sin\phi & \frac{l}{2}\cos\phi \end{bmatrix} \begin{bmatrix} -U_1 \sin\phi \\ U_1 \cos\phi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{l}{2}\sin\phi & \frac{l}{2}\cos\phi \end{bmatrix} \begin{bmatrix} -U_2 \sin\phi \\ U_2 \cos\phi \end{bmatrix}$$

$$Q_j = \begin{bmatrix} -(U_1 + U_2)\sin\phi \\ (U_1 + U_2)\cos\phi \\ (U_1 - U_2)(0.5\,l) \end{bmatrix}$$

forces

moment

$(x$

$$\begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$$

④ Euler-lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$\mathcal{L} = 0.5\left[m(\dot{x}^2 + \dot{y}^2) + I\dot{\phi}^2\right] - mgy$$

$$Q_j = \begin{bmatrix} -(U_1+U_2)\sin\phi \\ (U_1+U_2)\cos\phi \\ (U_1-U_2)(0.5\ell) \end{bmatrix}$$

① $$\frac{d}{dt}(m\dot{x}) - 0 = -(U_1+U_2)\sin\phi$$

$$\ddot{x} = -\frac{(U_1+U_2)}{m}\sin\phi$$

② $$\frac{d}{dt}(m\dot{y}) - (-mg) = (U_1+U_2)\cos\phi$$

$$\ddot{y} = -g + \frac{(U_1+U_2)}{m}\cos\phi$$

③ $$\frac{d}{dt}(I\dot{\phi}) - 0 = (U_1-U_2)(0.5\ell)$$

$$\ddot{\phi} = \frac{(U_1-U_2)(0.5\ell)}{I}$$

$U_1 + U_2 = U_s$

$U_1 - U_2 = U_d$

$$\boxed{\ddot{x} = -\frac{U_s}{m}\sin\phi \; ; \quad \ddot{y} = -g + \frac{U_s}{m}\cos\phi \; ; \quad \ddot{\phi} = \frac{0.5\,l}{I}\,U_d}$$

① x, y, ϕ $\quad$ dof = 3

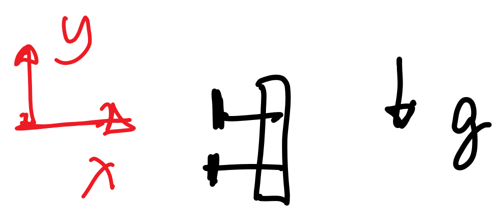
$\quad U_s, U_d$ $\quad$ actuators = 2 $\Big\}$ under-actuated

② (i) $\phi = 0$

$$\ddot{x} = 0 \; ; \quad \ddot{y} = -g + \frac{U_s}{m} \; ;$$

$$\ddot{y} = 0 \qquad U_s = mg$$

(ii) $\phi = 90^\circ$



$$\ddot{x} = -\frac{U_s}{m} \; ; \quad \ddot{y} = -g$$

✓ ✓

(iii) $0 < \phi < \pi/2$ $\phi = \pi/4$ $\cos\phi = \sin\phi = 1/\sqrt{2}$

$\ddot{x} = -\dfrac{U_s}{\sqrt{2}m}$ $\ddot{y} = -g + \dfrac{U_s}{\sqrt{2}m}$; $\ddot{\phi} = \dfrac{0.5l}{I} U_d$

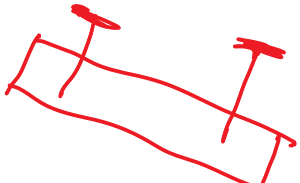
$\ddot{y} = 0$ $U_s \simeq \sqrt{2}mg$

$\ddot{x} = -g$

(iv) $\phi = -\pi/4$; $\ddot{x} = \dfrac{U_s}{\sqrt{2}m}$ $\ddot{y} = -g + \dfrac{U_s}{\sqrt{2}m}$

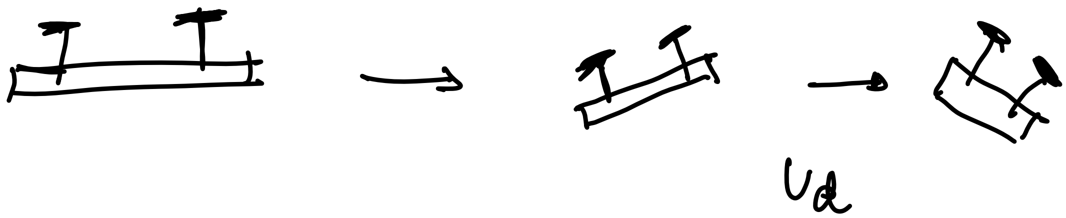
$\ddot{y} = 0$ $U_s = \sqrt{2}mg$

$\ddot{x} = +g$



$\ddot{x} = +g$

$\ddot{\phi} = \left(\dfrac{0.5l}{I}\right) U_d$ $U_d \rightarrow +\phi \longleftrightarrow -\phi$



① Hovering (HW)

② Tracking (Now)

$$U = U_0 + \delta u$$

nominal feedback

<u>Nominal</u>

$$m\ddot{x} = -U_s \sin\phi$$

$$m\ddot{y} = -mg + U_s \cos\phi$$

$$I\ddot{\phi} = 0.5\, l\, U_d$$

$$\ddot{x} = \ddot{y} = \ddot{\phi} = \dot{x} = \dot{y} = \dot{\phi} = \phi = 0$$

$\Rightarrow \quad 0 = -U_{s0} \sin(0)$ ✓

$\Rightarrow \quad 0 = -mg + U_{s0}(1) \quad \Rightarrow \boxed{U_{s0} = mg}$

$\Rightarrow \quad 0 = 0.5\, l \quad \Rightarrow \boxed{U_{d0} = 0}$

Feedback

$$m\ddot{x} = -U_s \sin\phi$$

$$m\ddot{y} = -mg + U_s \cos\phi$$

$$I\ddot{\phi} = 0.5\, l\, U_d$$

$$U_s = \underbrace{U_{so}}_{mg} + \delta U_{so} \; ; \; U_d = \cancel{U_{do}}^{\,0} + \delta U_{do}$$

$$m\ddot{x} = -(mg + \delta U_s)\sin\phi$$

$$= -mg \underbrace{\sin\phi}_{\phi} - \underbrace{\delta U_s \;(\phi)}_{\text{2 small terms multiplied}}$$

$$\Rightarrow \quad \ddot{x} = -g\phi \qquad (\text{Linearized eq}^n)$$

$$m\ddot{y} = -mg + (mg + \delta U_{so})\cancel{\cos\phi}^{\,1}$$

$$m\ddot{y} = \cancel{-mg} + \cancel{mg} + \delta U_{so}$$

$$\Rightarrow \quad \ddot{y} = \frac{\delta U_{so}}{m}$$

$$I\ddot{\phi} = 0.5\,l\,(\cancel{u_{d_0}} + \delta u_{d_0})$$

$$\Rightarrow \quad \ddot{\phi} = \frac{0.5\,l}{I}\,\delta u_d$$

$$\ddot{x} = -g\phi$$

$$\ddot{y} = \frac{\delta u_s}{m}$$

$$\ddot{\phi} = \frac{0.5\,l}{I}\,\delta u_d$$

Feedback linearization

$$\phi_{ref} = -\frac{1}{g}\left(\ddot{x}_{ref} + k_{px}(x_{ref} - x) + k_{dx}(\dot{x}_{ref} - \dot{x})\right)$$

$$\delta u_s = m\left(\ddot{y}_{ref} + k_{py}(y_{ref} - y) + k_{dy}(\dot{y}_{ref} - \dot{y})\right)$$

$$\delta u_d = -k_{d\phi}\dot{\phi} + k_{p\phi}(\phi_{ref} - \phi)$$

$$u_s = u_{s_0} + \delta u_s = mg + \delta u_s$$

$$u_s = u_{s_0} + \delta u_s = mg + \delta u_s$$

$$u_d = 0 + \delta u_d$$