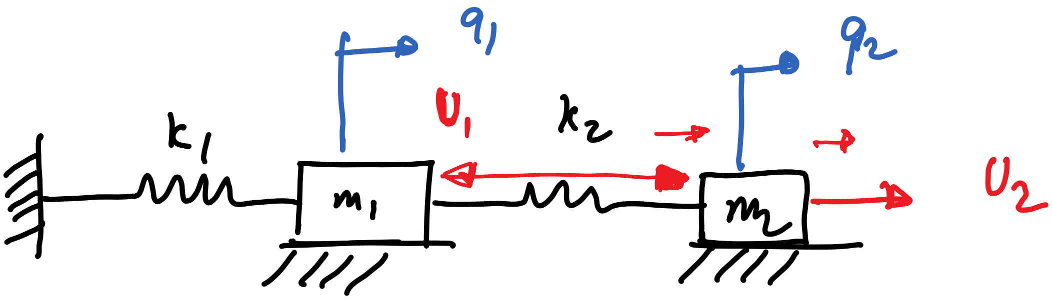
Linear Control

system is linear in state (control)



m_1, m_2 — masses

$k_1\ k_2$ — spring constant

u_1, u_2 — controls (forces)

q_1, q_2 — mass displacements

$$T = 0.5\, m_1 \dot{q}_1^2 + 0.5\, m_2 \dot{q}_2^2$$

$$V = 0.5\, k_1\, q_1^2 + 0.5\, k_2 (q_1 - q_2)^2$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = 0.5\, m_1 \dot{q}_1^2 + 0.5\, m_2 \dot{q}_2^2 + 0.5\, k_1 q_1^2 + 0.5\, k_2 (q_1 - q_2)^2$$

$$\mathcal{L} = 0.5\, \underline{m_1 \dot{q}_1^2} + 0.5\, m_2 \dot{q}_2^2 + 0.5\, k_1 q_1^2 + 0.5\, k_2 (q_1 - q_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \qquad j = 1, 2$$

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① q_1: $$\frac{d}{dt}\left(0.5\,(2 m_1 \dot{q}_1)\right) + 0.5\,(2 k_1 q_1) + 0.5\,(2 k_2)(q_1 - q_2) = -u_1$$

$$m_1 \ddot{q}_1 + k_1 q_1 + k_2 (q_1 - q_2) = -u_1$$

$$\ddot{q}_1 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) q_1 + \frac{k_2}{m_1} q_2 - \frac{u_1}{m_1} \quad \text{—} ①$$

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② q_2: $$\frac{d}{dt}\left(0.5\,(2 m_2 \dot{q}_2)\right) + 0.5\,(2 k_1)(q_1 - q_2)(-1) = u_1 + u_2$$

$$m_2 \ddot{q}_2 + k_1 (q_2 - q_1) = u_1 + u_2$$

$$\ddot{q}_2 = \frac{k_2}{m_2} q_1 - \frac{k_2}{m_2} q_2 + \frac{U_1}{m_2} + \frac{U_2}{m_2} \quad -(2)$$

State space representation

$$x_1 = q_1$$

$$x_2 = q_2$$

$$x_3 = \dot{q}_1$$

$$x_4 = \dot{q}_2$$

$$\rightarrow \quad \dot{x}_1 = \dot{q}_1 = x_3$$

$$\dot{x}_2 = \dot{q}_2 = x_4$$

from (1)

$$\dot{x}_3 = \ddot{q}_1 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) x_1 + \left(\frac{k_2}{m_1}\right) x_2 - \frac{U_1}{m_1}$$

$$\dot{x}_4 = \ddot{q}_2 = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_2 + \frac{U_1}{m_1} + \frac{U_2}{m_2}$$

from (2)

$$X = [x_1, x_2, x_3, x_4]$$

$$\underset{4\times1}{\dot{x}} = \underset{4\times4}{A}\,\underset{4\times1}{x} + \underset{4\times2}{B}\,\underset{2\times1}{u}$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underset{4\times4}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underset{4\times2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{m_1} & 0 \\ \frac{1}{m_2} & \frac{1}{m_2} \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\text{state-space equation}}$$

Check stability of uncontrolled system

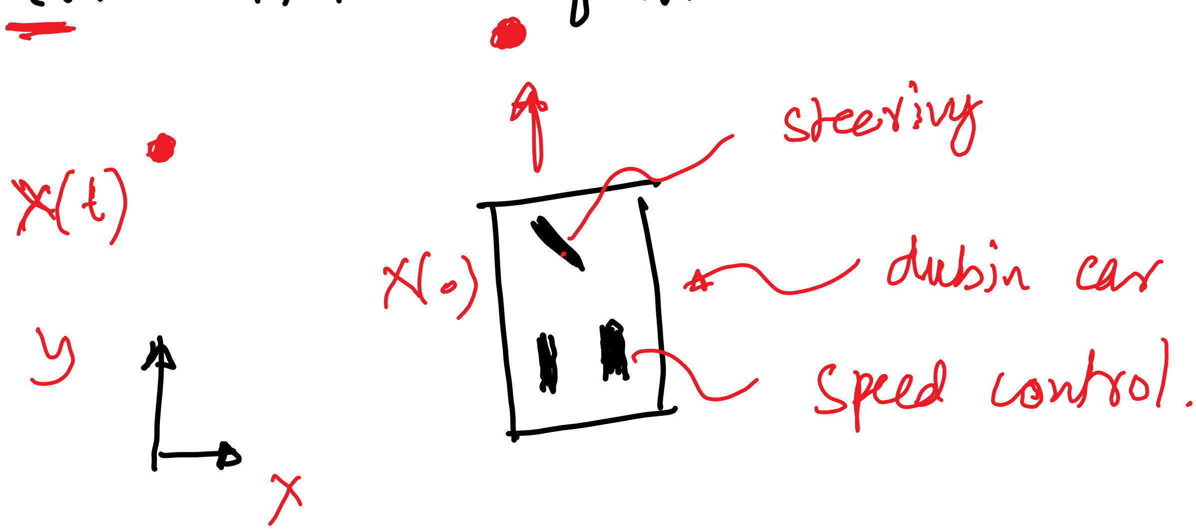
$$\dot{x} = Ax \quad \text{— uncontrolled system}$$

Compute eigenvalues and check if all the real parts are negative. If they are negative then the system is stable

Use control (u) to either stabilize the system or make it more stable

However, one needs to check if the system is controllable

Controllability: A linear system is controllable if and only if it can be transferred from any initial state $x(0)$ to any terminal state $x(t)$ within a finite time



$\Rightarrow$ $C_o = [A^{n-1}B, A^{n-2}B, A^{n-3}B, \dots AB, B]$

n = system dof e.g. spring-mass $n=4$

$\Rightarrow$ $\text{rank}(C_o) = n$ system is controllable

$\text{rank}(C_o) < n$ system is not controllable

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pip install control

import control
Co = control.ctrb(A,B)
np.linalg.matrix_rank(Co)
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Methods of control

① Pole placement

Assume $u = -Kx$ $\quad$ K = gain matrix

$$\dot{x} = Ax + Bu$$

$$= Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{place eigenvalues at a certain location}} x$$

place poles at "p" ← user choice

K = control.place (A, B, p) ← user choice

② Linear quadratic controller

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→ $\dot{x} = Ax + Bu$

Compute u such that it minimizes

$$J = \underbrace{x^T(t_f)\,F\,x(t_f)}_{\text{terminal cost}} + \ldots$$

$$\int_0^{t_f} \left(x^T Q x + u^T R u + 2x^T N u\right) dt$$

F, Q, R, N — user chosen matrices

relative

$Q \gg R$ aggressive control

$Q \ll R$ less aggressive control

$Q >$ positive definite — eign val > 0

$R \geq$ semi-positive definite — eign val ≥ 0

$u = -Kx$

gain K

?

$K = R^{-1}(B^T P + N^T)$ $\{u = -Kx\}$

$-\dot{P} = A^T P + PA - (PB + N) R^{-1} (B^T P + N^T) + Q = 0$

Ricatti differential equation

$P(t_f) = F(t_f)$

Special case

$$J = \int_0^\infty (x^T Q x + u^T R u + 2 x^T N u) dt$$

infinite horizon problem

$u = -Kx$ & $K = -R^{-1}(B^T P + N^T)$

$$AP + PA - (PB + N) R^{-1} (B^T P + N^T) + Q = 0$$

Steady state Ricatti equation

K, P, E = control.lqr (A, B, Q, R, N)

gain

solution to Ricatti equation

eigenvalues of closed loop: eig (A-BK)

Linear control for a non-linear system

$$\dot{x} = f(x, u) \qquad -(1)$$

We will linearize this system about an operating point (x_0, u_0) and use the linearized system for control.

$$x \rightarrow x_0 + \delta x \quad ; \quad u \rightarrow u_0 + \delta u$$

Putting this in (1)

Taylor series.

$$\dot{x}_0 + \delta\dot{x} = f(x_0 + \delta x, u_0 + \delta u)$$

$$\dot{x}_0 + \delta\dot{x} = f(x_0, u_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial u}(u - u_0)$$

$$\dot{x}_0 = f(x_0, u_0) \qquad + \text{higher order terms}$$

$$\delta\dot{x} = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial u}\,\delta u$$

$$\boxed{\begin{aligned}\delta \dot{x} &= \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial u}\,\delta u \\ \delta \dot{x} &= A\,\delta x + B\,\delta u\end{aligned}}$$

where $A = \frac{\partial f}{\partial x}$; $B = \frac{\partial f}{\partial u}$

1. use lqr / pole placement to compute

$$\delta u = -K\,\delta x$$

2. Note the control is

$$u = u_0 + \delta u$$

Example:

Differential drive car

$$\dot{x} = v \cos\theta$$

$$\dot{y} = v \sin\theta$$

$$\dot{\theta} = \omega$$

$$X = [x, y, \theta]^T; \quad u = [v, \omega]$$

Assume an operating point X_0, u_0

Assume an operating point x_0, u_0

(x_0, y_0, θ_0) $\rightarrow$ (v_0, ω_0)

$$\delta\dot{x} = A\,\delta x + B\,\delta u$$

$$\bar{B}(\theta)\begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$f = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x}(v\cos\theta) & \frac{\partial(v\cos\theta)}{\partial y} & \frac{\partial(v\cos\theta)}{\partial \theta} \\ \frac{\partial(v\sin\theta)}{\partial x} & \frac{\partial(v\sin\theta)}{\partial y} & \frac{\partial(v\sin\theta)}{\partial \theta} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -v\sin\theta \\ 0 & 0 & v\cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

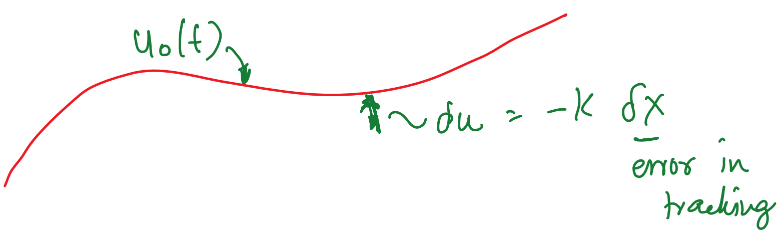
$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial v\cos\theta}{\partial v} & \frac{\partial v\cos\theta}{\partial \omega} \\ \frac{\partial v\sin\theta}{\partial v} & \frac{\partial v\sin\theta}{\partial \omega} \\ \frac{\partial \omega}{\partial v} & \frac{\partial \omega}{\partial \omega} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$$

$$u \to (v, \omega)$$

$$\delta\dot{x} = \begin{bmatrix} 0 & 0 & -v_0\sin\theta_0 \\ 0 & 0 & v_0\cos\theta_0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix} \delta w$$

$u = u_0 + \delta u_0 \rightarrow$ pole placement/LQR (feedback)

$u_0 \rightarrow$ found using trajectory optimization (open-loop)



This controller is different from what we did previously

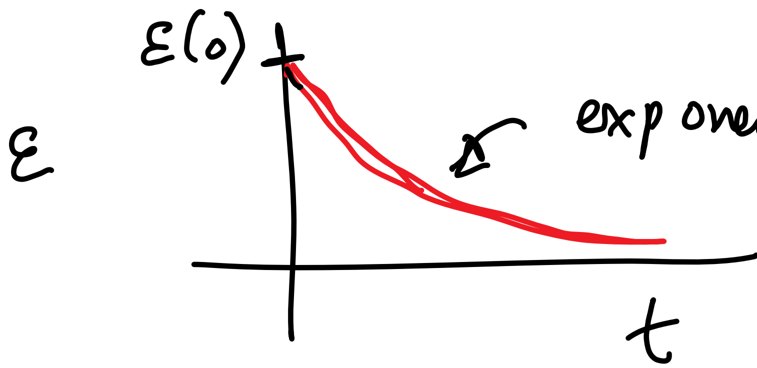
$$\dot{x} = f(x, u) \quad - \text{car}$$

$$\dot{x} = \dot{x}_{ref} + k_p (x_{ref} - x) \quad - \text{controller for car}$$

[feedback linearization]

$$\underbrace{\dot{x} - \dot{x}_{ref}}_{\dot{\varepsilon}} = k_p \underbrace{(x_{ref} - x)}_{-\varepsilon}$$

$$\dot{\varepsilon} + k_p \varepsilon = 0 \quad \Rightarrow \quad \boxed{\varepsilon = A e^{-k_p t}}$$



exponentially goes to zero

Manipulator system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \bar{B}(q)u$$

$\bar{B}(q)$ — $n \times m$

u — $m \times 1$

$$\dot{x} = f(x,u)$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \begin{matrix} \text{— angle} \\ \text{— angular velocity.} \end{matrix}$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\{-G(q) - C(q,\dot{q})\dot{q} + B(q)u\} \end{bmatrix}$$

$$= f(x,u)$$

$$\dot{x} = f(x,u) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\tilde{A} = \frac{\partial f}{\partial x} \quad ; \quad \tilde{B} = \frac{\partial f}{\partial u}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\{-C(q,\dot{q})\dot{q} - G(q) + B(q)u\} \end{bmatrix}$$

$$x = [q \, , \, \dot{q}]$$

$$\hat{A} = \frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial q & \partial f_1/\partial \dot{q} \\ \partial f_2/\partial q & \partial f_2/\partial \dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} \overset{0}{\cancel{\partial \dot{q}/\partial q}} & \underline{\underline{\partial \dot{q}/\partial \dot{q}}} = I \\ \dfrac{\partial f_2}{\partial q} & \dfrac{\partial f_2}{\partial \dot{q}} \end{bmatrix}$$

$$\frac{\partial f_2}{\partial q} = \frac{\partial}{\partial q} M^{-1}(q)\left[-C(q,\dot q)\dot q - G(q) + B(q)u\right]$$

$M\ddot q \simeq 0$ at the operating point

$$= \frac{\partial M^{-1}(q)}{\partial q}\left[-C(q,\dot q)\dot q - G(q) + B(q)u\right]$$

$$+ M^{-1}(q)\left[-\frac{\partial C(q,\dot q)\dot q}{\partial q} - \frac{\partial G}{\partial q} + \frac{\partial}{\partial q}B(q)u\right]$$

$\frac{\partial C(q,\dot q)\dot q}{\partial q} = 0$, $\frac{\partial G}{\partial q} \neq 0$, $\frac{\partial}{\partial q}B(q)u \neq 0$

$\dot q = 0$ at the operation point

Trick: $MM^{-1} = I$

$$\frac{\partial M}{\partial q}M^{-1} + M\frac{\partial M^{-1}}{\partial q} = 0$$

$$M\frac{\partial M^{-1}}{\partial q} = -\frac{\partial M}{\partial q}M^{-1}$$

Pre-multiply with M^{-1}

$$\underbrace{M^{-1}M}_{I}\frac{\partial M^{-1}}{\partial q} = -M^{-1}\frac{\partial M}{\partial q}M^{-1}$$

$$\frac{\partial M^{-1}}{\partial q} = M^{-1}\left(\frac{\partial M}{\partial q}\right)M^{-1}$$

$$\frac{\partial f_2}{\partial q} = -M^{-1}(q)\frac{\partial G}{\partial q} + \sum_j M^{-1}\frac{\partial B_j}{\partial q}u_j$$

$$\frac{\partial f_2}{\partial \dot q} = \frac{\partial}{\partial \dot q} M^{-1}(q)\left[-\cancel{G(q)} - C(q,\dot q)\dot q + B(q)u\right]$$

0

0

$$\boxed{\tilde A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & I \\ -M^{-1}\frac{\partial G}{\partial q} + \sum_j M^{-1}\frac{\partial B_j}{\partial q}u_j & 0 \end{bmatrix}}$$

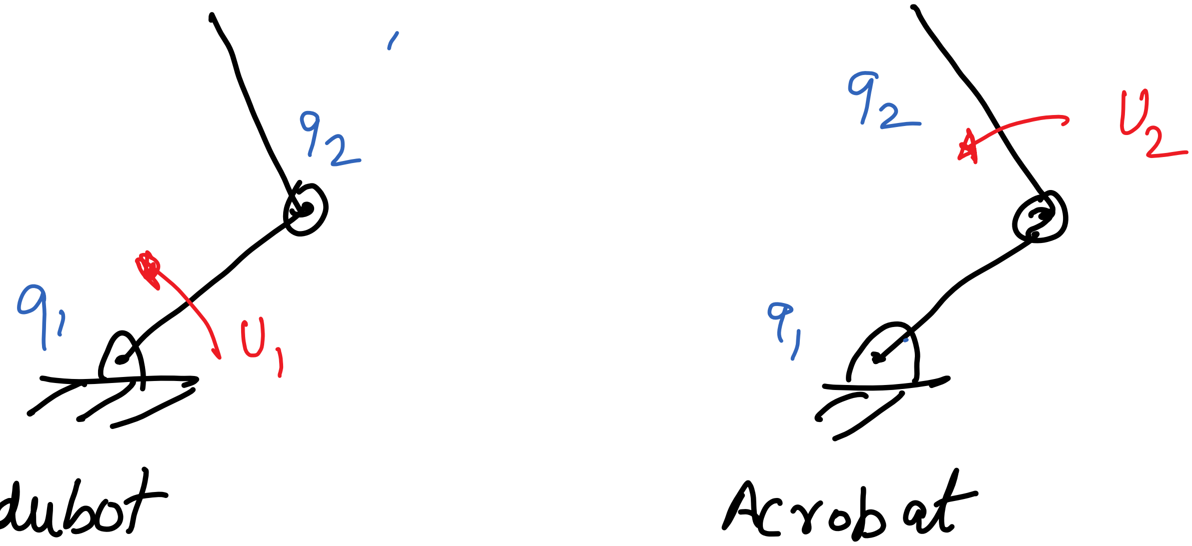
$$\tilde B = \frac{\partial f}{\partial u} =$$

$$f = \begin{bmatrix} \dot q \\ M^{-1}(q)\left(-C(q,\dot q)\dot q - G(q) + B(q)u\right) \end{bmatrix}$$

$$\boxed{\tilde B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ M^{-1}(q)B(q) \end{bmatrix}}$$

Example: Underactuated Double Pendulum

(1)



Degrees of freedom = 2 (q_1, q_2)

Actuators = 1 (U_1 for pendubot
U_2 for acrobot)

$$\Rightarrow \quad U_i = -k_1 q_1 - k_2 q_2 - k_3 \dot{q}_1 - k_4 \dot{q}_2$$

$i = 1$ or 2