Control of manipulators

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau$$

$$\downarrow$$

$$AX = b \qquad \ddot{\theta} = \ddot{q}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

$$\underbrace{M(q)}_{A}\,\underbrace{\ddot{q}}_{X} = \underbrace{\tau - C(q,\dot{q})\dot{q} - G(q)}_{b}$$

$M(q)$ = mass/inertia

$C(q,\dot{q})\dot{q}$ = coriolis acceleration/torque

$G(q)$ = gravitational torque

τ = external torque

a) Set-pt. control :

b) Trajectory control :

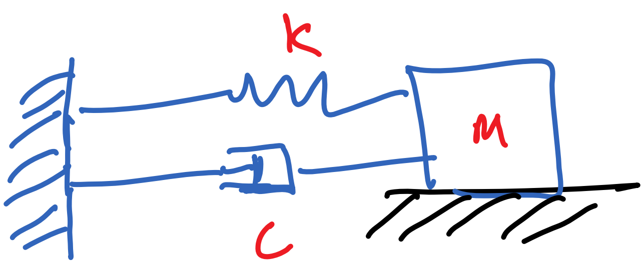
Simplest example:

$$M(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = \tau$$

$$\downarrow$$

$$m\ddot{q} + c\dot{q} + kq = \tau$$

spring-mass-damper system



$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = \frac{\cancel{\tau}}{\cancel{m}}\,0$$

free vibration

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \;;\quad 2\xi\omega_n = \frac{c}{m}$$

$$\downarrow$$

$$\boxed{\xi = \frac{c}{2\sqrt{mk}}}$$

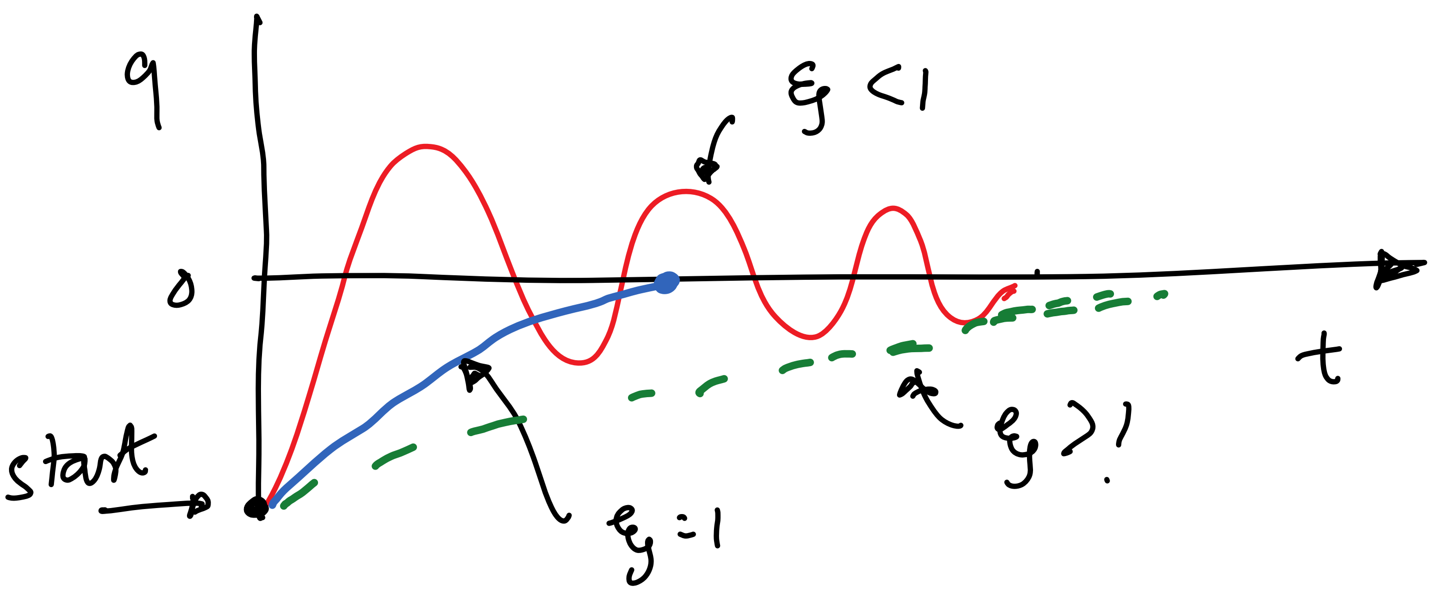
3 cases

$$\xi = \frac{c}{2\sqrt{mk}}$$

① $\xi > 1 \Rightarrow c > 2\sqrt{mk}$ overdamped

② $\xi = 1 \Rightarrow c = 2\sqrt{mk}$ critically damped

③ $\xi < 1 \Rightarrow c < 2\sqrt{mk}$ under-damped



fastest decay to zero

we will use Force/torque to achieve $\xi = 1$ to achieve fastest damping.

$$m\ddot{q} + c\dot{q} + kq = \tau \quad — (1)$$

$$\tau = -k_p q - k_d \dot{q} \quad — (2)$$

proportional control

derivative control

(PD control)

Put (2) in (1)

designer's choice

$$m\ddot{q} + c\dot{q} + kq = -k_p q - k_d \dot{q}$$

$$m\ddot{q} + (c + k_d)\dot{q} + (k + k_p) q = 0$$

$$(c + k_d) = 2\sqrt{m(k + k_p)}$$

critically damped

Solve for k_d

$$\Rightarrow c^2 + 2k_d c + k_d^2 = 4mk + 4mk_p$$

$$\Rightarrow k_d^2 + 2k_d c + (c^2 - 4mk - 4mk_p) = 0$$

Solve for k_d

$$k_d = \frac{-2 \pm \sqrt{(2c)^2 - 4(1)(c^2 - 4(k+k_p)m)}}{2(1)}$$

Simplify:

$$k_d = -c \pm 2\sqrt{(k+k_p)m}$$

$\downarrow$

Choose stable gain

$$\boxed{k_d = -c + 2\sqrt{(k+k_p)m}}$$

$k_p, k_d \rightarrow$ designer's choice

Extend to 2-D system

1D $\quad m\ddot{q} + c\dot{q} + kq = \tau$ $\quad \| \; \tau = \underbrace{-k_p q - k_d \dot{q}}^{2}$

1 equation

2D

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\tau = -k_p q - k_d \dot{q}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = -\underbrace{\begin{bmatrix} k_{p11} & k_{p12} \\ k_{p21} & k_{p22} \end{bmatrix}}_{4} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \underbrace{\begin{bmatrix} k_{d11} & k_{d12} \\ k_{d21} & k_{d22} \end{bmatrix}}_{4} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

8 free parameters

2 conditions (critical damping)

Feedback Linearization / control partitioning

① $M\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$

② choose $\tau = \hat{M}(-K_p q - K_d \dot{q}) + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q)$

$\hat{M}, \hat{C}, \hat{G} \rightarrow$ estimates of M, C, G

Lets assume $M = \hat{M}$; $C = \hat{C}$; $G = \hat{G}$

Substitute ② in ①

$$M\ddot{q} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{G(q)} = M(-K_p q - K_d \dot{q}) + \cancel{C(q,\dot{q})\dot{q}} + \cancel{G(q)}$$

$$M(\ddot{q} + K_d\dot{q} + K_p q) = 0$$

Since $M \neq 0$ $\quad \ddot{q} + K_d\dot{q} + K_p q = 0$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} K_{d_1} & 0 & 0 & \cdots \\ 0 & K_{d_2} & 0 & \cdots \\ 0 & 0 & K_{d_3} & \cdots \\ \cdots & & & K_{d_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} K_{p_1} & 0 & 0 & \cdots \\ 0 & K_{p_2} & 0 & \cdots \\ 0 & 0 & K_{p_3} & \cdots \\ 0 & 0 & \cdots & K_{p_n} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$\ddot{q}_1 + k_{d_1}\dot{q}_1 + k_{p_1} q_1 = 0$$

$$\ddot{q}_2 + k_{d_2}\dot{q}_2 + k_{d_2} q_2 = 0$$

$$\vdots$$

$$\ddot{q}_n + k_{d_n}\dot{q}_n + k_{p_n} q_n = 0$$

n equations

$$m\ddot{q} + c\dot{q} + kq = \tau$$

$$m\ddot{q} + (c + k_d)\dot{q} + (k + k_p)q = 0$$

$$\boxed{k_d = -c + 2\sqrt{m(k + k_p)}}$$

$$c = 0 \;;\; m = 1 \;;\; k = 0$$

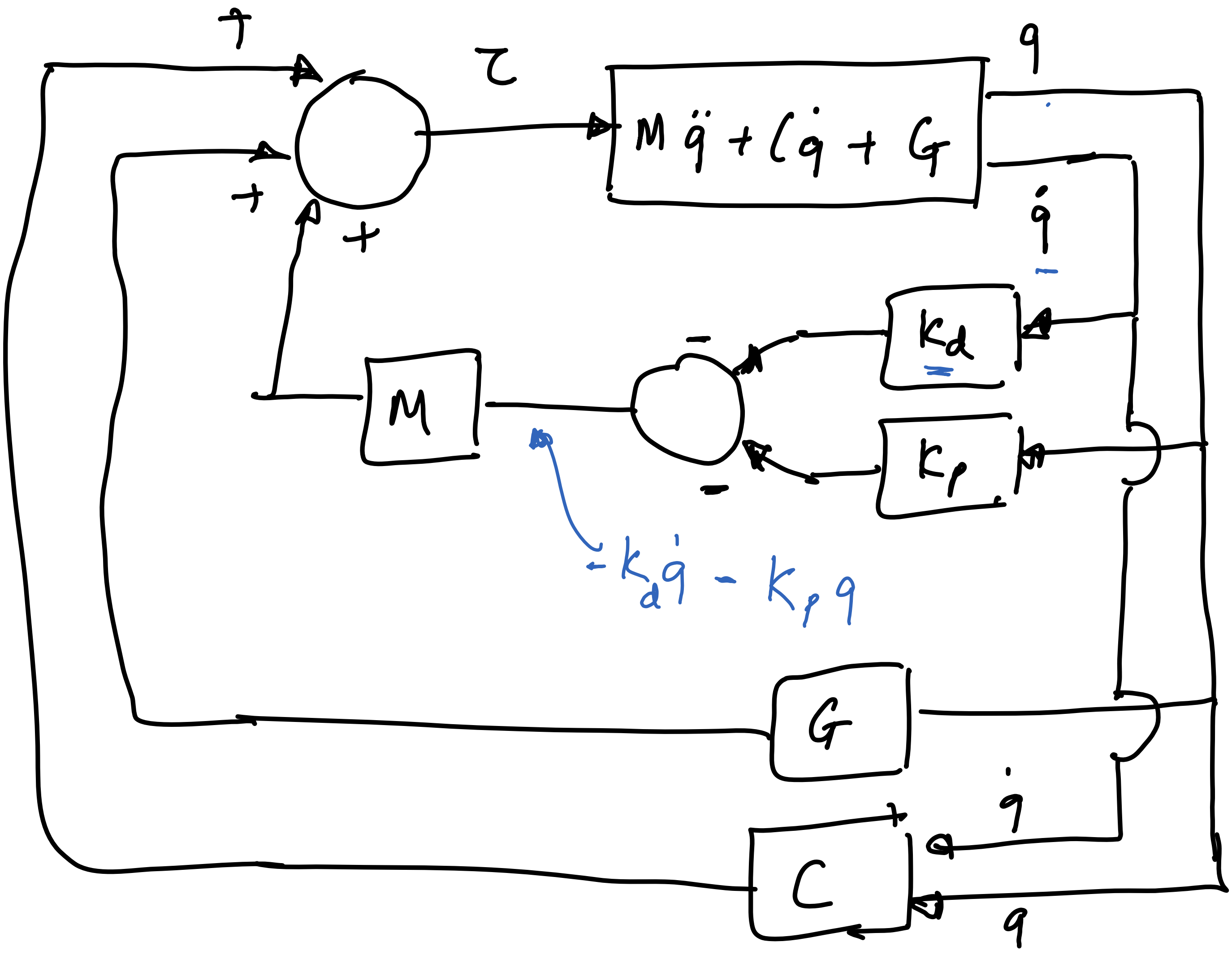
$$k_{d_i} = 0 + 2\sqrt{(1)(0 + k_{p_i})}$$

$$\boxed{k_{d_i} = 2\sqrt{k_{p_i}}}$$

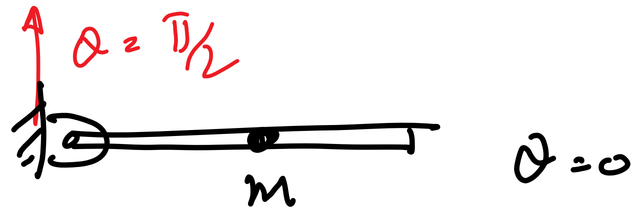
① - $M\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$

② - Choose $\tau = \hat{M}(-k_p q - k_d \dot{q}) + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q)$

Block diagram



Example: 1-Link pendulum



Start: $\theta = 0$

End: $\theta = \pi/2$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

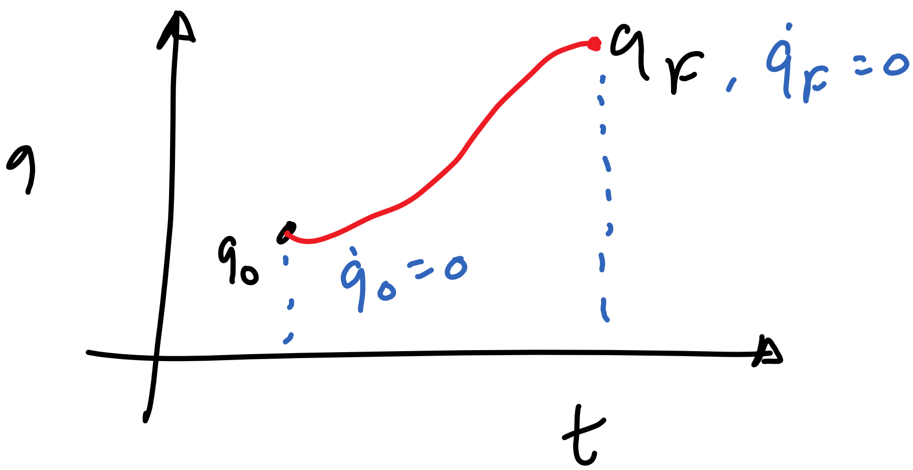
$$ml^2\ddot{\theta} + 0 + mgl\sin(\theta) = \tau$$

Controllers

① $\tau = -k_p(q - q_d) - k_d\dot{q}$ $\quad$ q_d = reference $= \pi/2$

② $$\tau = \underbrace{M}_{ml^2}(-k_p(q-q_d) - k_d\dot{q}) + \underbrace{C(q,\dot{q})}_{=0}\dot{q} + \underbrace{G(q)}_{= mgl\sin\theta}$$

Feedback linearization for trajectory tracking



$$q_{ref}(t) = a_0 + a_1 t + a_2 t + a_3 t^3$$

Dynamics: $\underline{M\ddot{q} + C\dot{q} + G} = \tau$

Control: $\tau = M\left(\underline{\ddot{q}_{ref} - K_d(\dot{q} - \dot{q}_{ref}) - K_p(q - q_{ref})}\right) + G + C\dot{q}$

$$M\ddot{q} = M(\underline{\ddot{q}_{ref} - K_d(\dot{q} - q_{ref})} - K_p(q - q_{ref}))$$

$$M(\underbrace{\ddot{q} - \ddot{q}_{ref}}_{\ddot{e}_{ref}} + K_d(\underbrace{\dot{q} - \dot{q}_{ref}}_{\dot{e}_{ref}}) + K_p(\underbrace{q - q_{ref}}_{e}) = 0$$

$\dot{e}$ ref $\quad\quad$ $\dot{e}$ ref $\quad\quad$ e

$$\underline{M}\,\ddot{q} = M\big(\underline{\ddot{q}_{ref} - k_d(\dot{q} - \dot{q}_{ref})} - k_p(q - q_{ref})\big)$$

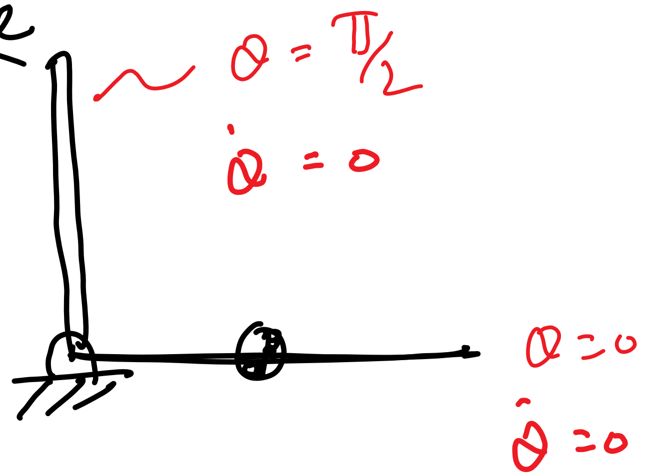
$$M(\underbrace{\ddot{q} - \ddot{q}_{ref}}_{\ddot{e}}) + k_d(\underbrace{\dot{q} - \dot{q}_{ref}}_{\dot{e}}) + k_p(\underbrace{q - q_{ref}}_{e}) = 0$$

$$M\ddot{e} + k_d\dot{e} + k_p e = 0$$

spring-mass-damper

Block diagram

Example



$$\theta_{ref}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Feedback Linearization in Task space

- So far q_{ref}, $\dot{q}_{ref}$, $\ddot{q}_{ref}$ — q = joint position
- $\tau = M(\ddot{q}_{ref} - k_d(\dot{q} - \dot{q}_{ref}) - k_p(q - q_{ref}) + C + G$ — IV

- What if $x_{ref}, \dot{x}_{ref}, \ddot{x}_{ref}$ / $y_{ref}, \dot{y}_{ref}, \ddot{y}_{ref}$ } same ideas to trajector

- $\tau = ?$

$\begin{Bmatrix} x \\ y \end{Bmatrix} \Rightarrow X = f(q)$

forward kinematics

$X_{ref} = f(q_{ref})$

$$\boxed{q_{ref} = f^{-1}(X_{ref})}$$

Inverse kinematics (using fsolve)

I

$$\underline{\dot{x}} = \underbrace{\frac{\partial f}{\partial q}}_{=J} \dot{q} \qquad \dot{x}_{ref} = J \dot{q}_{ref}$$

$$\boxed{\dot{q}_{ref} = J^{-1} \dot{x}_{ref}} \quad — \text{I}$$

$$\dot{x} = J \dot{q}$$

$$\ddot{x} = \frac{dJ}{dt} \dot{q} + J \ddot{q}$$

$$\ddot{x} = \dot{J} \dot{q} + J \ddot{q}$$

$$\ddot{x}_{ref} = \dot{J} \dot{q}_{ref} + J \ddot{q}_{ref}$$

$$\boxed{\ddot{q}_{ref} = J^{-1}\left(\ddot{x}_{ref} - \dot{J} \dot{q}_{ref}\right)} \quad — \text{II}$$

Given $\ddot{x}_{ref}, \dot{x}_{ref}, x_{ref}$. Compute $\ddot{q}_{ref}, \dot{q}_{ref}, q_{ref}$ using I, II, III

τ from IV