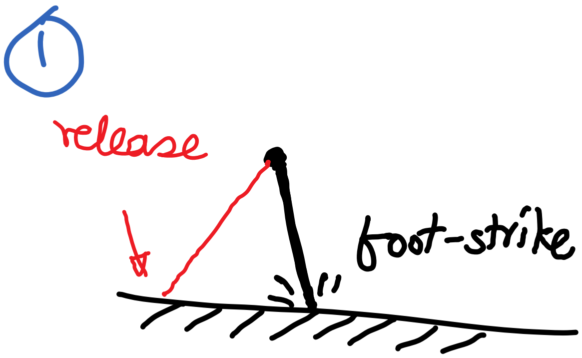
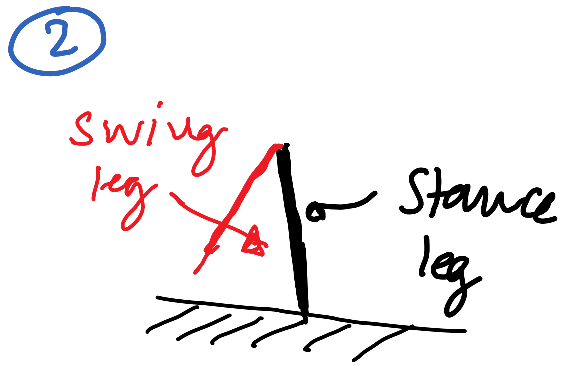
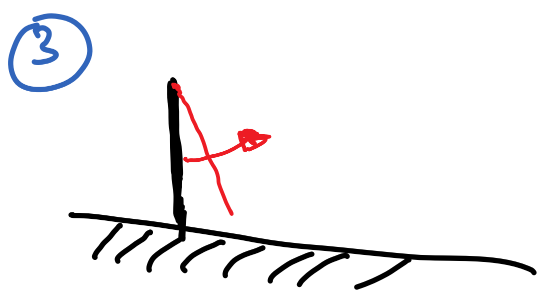
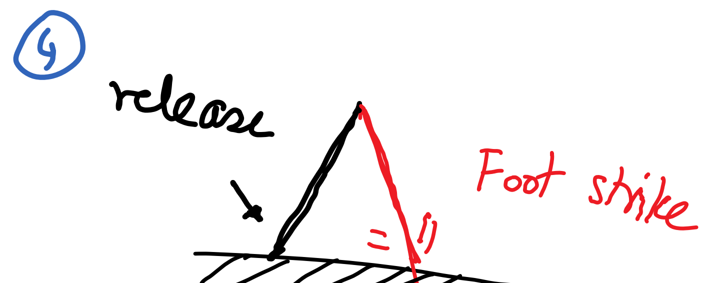
Passive dynamic Walking









1 → 2 → 3 → 4

1 = 4

one-step

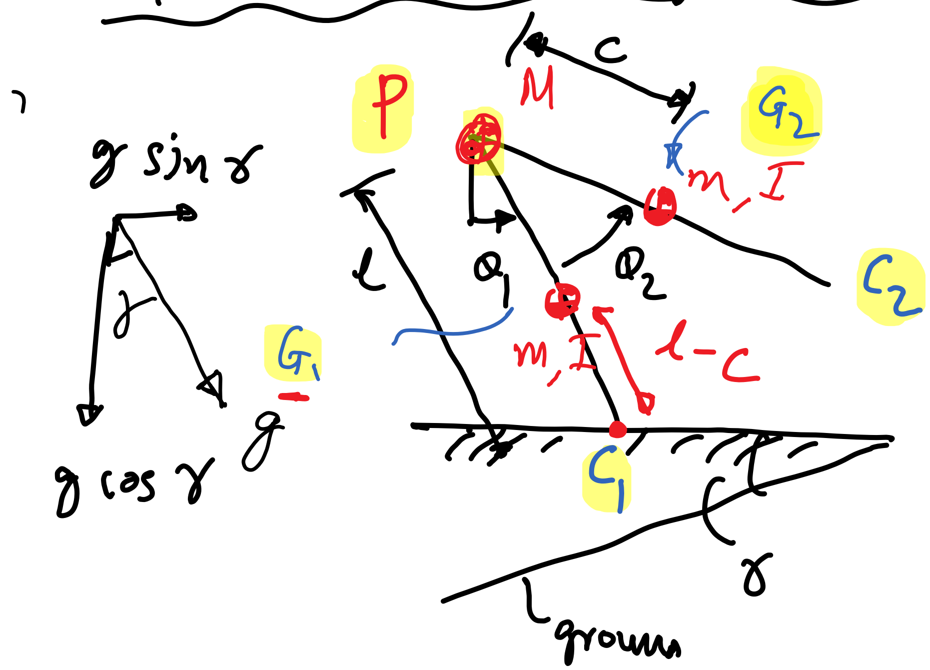
footstrike

→ single-stance $\xrightarrow[\text{ground}]{\text{detect}}$ footstrike → single stance

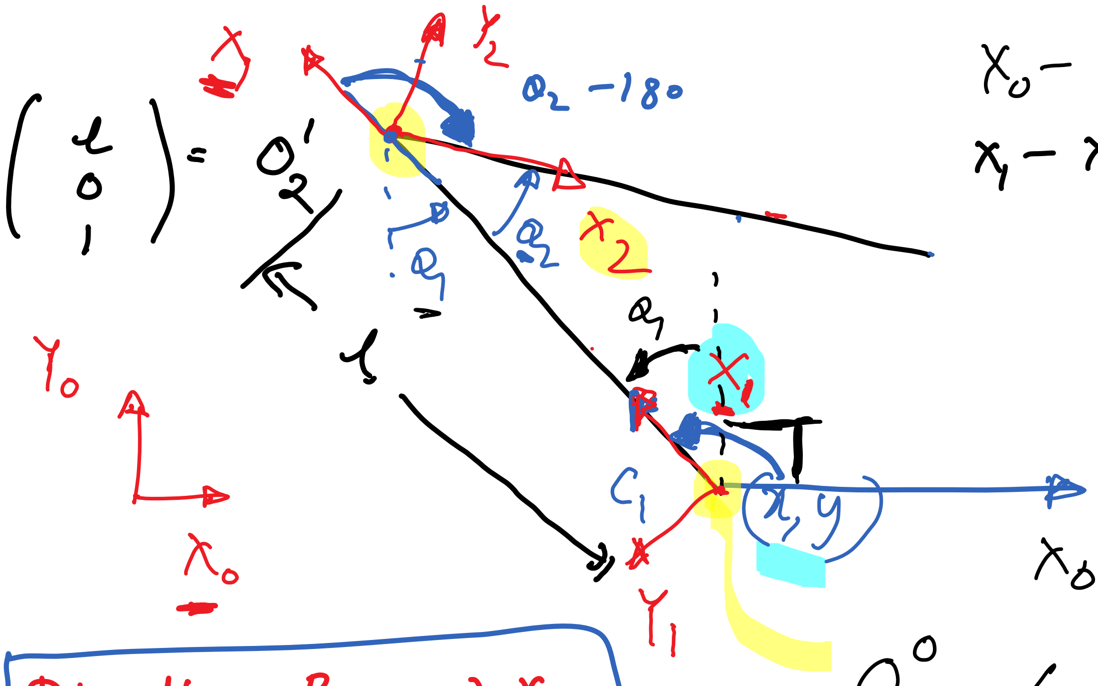
single-stance: 1 detect ground: 3 footstrike: 2

one-step

① Equations of single stance



a) Positions / Velocities



$X_0 - X_1 = 90 + \theta_1$

$X_1 - X_2 = \theta_2 - 180$

$O_1^0 \doteq \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$\boxed{\text{Floating Base} \Rightarrow x, y \text{ changes every step}}$

$$R_1^0 = \begin{bmatrix} \cos(90+\theta_1) & -\sin(90+\theta_1) \\ \sin(90+\theta_1) & \cos(90+\theta_1) \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} \cos(\theta_2-180) & -\sin(\theta_2-180) \\ \sin(\theta_2-180) & \cos(\theta_2-180) \end{bmatrix}$$

$$O_1^0 = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad O_2^1 = \begin{pmatrix} \ell \\ 0 \\ 1 \end{pmatrix}$$

$$C_1^0 = O_1^0 \ ; \quad G_1^0 = H_1^0 \, G_1^1$$

$$= \begin{pmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell - c \\ 0 \\ 1 \end{pmatrix}$$

$$P^0 = H_1^0 P^1 = \begin{pmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell \\ 0 \\ 1 \end{pmatrix}$$

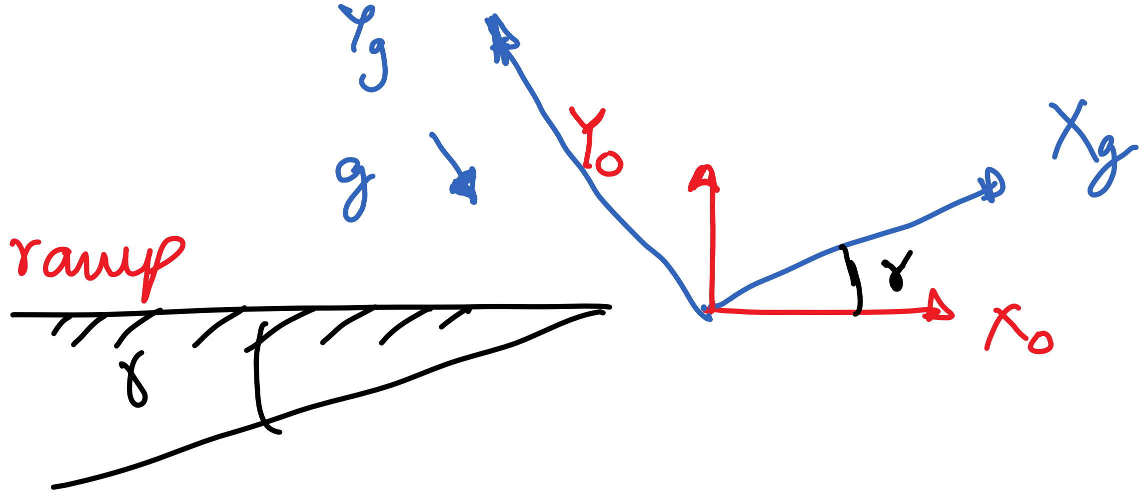
$$G_2^0 = H_2^0 \, G_2^2$$

$$= H_1^0 \, H_2^1 \, G_2^2$$

$$= \begin{pmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_2^1 & o_2^1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$$

$$C_2^0 = H_2^0 \, C_2^2$$

$$= \begin{pmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_2^1 & o_2^1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ 0 \\ 1 \end{pmatrix}$$



$$r^0 = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} r^g$$

$$r^0 = R(\gamma)\, r^g$$

$$r^g = R^{-1}(\gamma)\, r^0$$

$$\underline{r^g} = R^T\underline{(\gamma)}\, r^0$$

as $R^T R = I$

$R^{-1} = R^T$

$$r^g = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) \\ -\sin(\gamma) & \cos(\gamma) \end{bmatrix} r^0$$

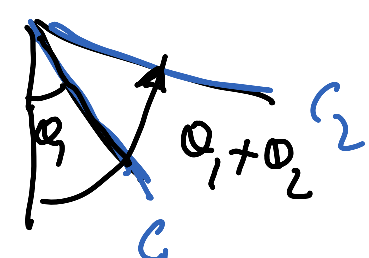
$$V_* = J_* \dot{q} = \frac{\partial r_*}{\partial q} \dot{q}$$

$C_1^0, G_1^0, \vdots$

$$* = C_1, C_2, G_1, G_2, P$$

(b) Euler-lagrange Equations

$$T = \frac{1}{2} M \left(V_{P_x}^2 + V_{P_y}^2 \right) + \frac{1}{2} m \left(V_{G_1 x}^2 + V_{G_1 y}^2 \right) + \frac{1}{2} m \left(V_{G_2 x}^2 + V_{G_2 y}^2 \right) + \frac{1}{2} I \dot{\theta}_1^2 + \frac{1}{2} I \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2$$



$$V = M g\, y_P^g + m g\, y_{G_1}^g + m g\, y_{G_2}^g$$

$$P^g = R^T(\gamma)\, P^0 \qquad R^T(\gamma) G_1^0 \qquad R^T(\gamma) G_2^0$$

only y- component

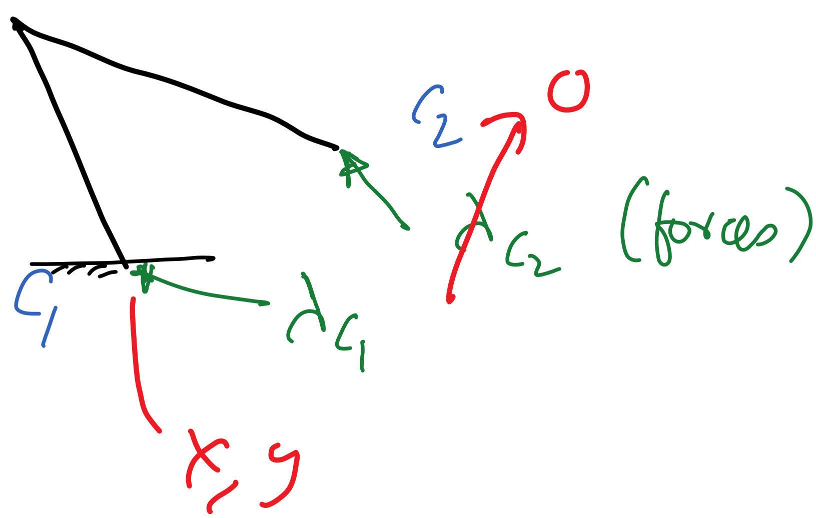
3) Euler-lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \lambda_{c_1}\overset{J_{c_1}}{\overset{\shortparallel}{\left(\frac{\partial \gamma_{c_1}}{\partial q}\right)}} + \overset{0}{\lambda_{c_2}}\overset{J_{c_2}}{\overset{\shortparallel}{\left(\frac{\partial \gamma_{c_2}}{\partial q}\right)}}$$

$$\mathcal{L} = T - V \qquad\qquad q = \{\underline{\underline{x, y}}, \theta_1, \theta_2\}$$

$\lambda_{c_1}, \lambda_{c_2} \Rightarrow$ lagrange multipliers (unknowns)

Constraint forces



$$\ddot{x} = 0 \; ; \; \ddot{y} = 0$$

In E-L equation,

1^{st} 2 equation $\Rightarrow \lambda_{c_1} = \lambda_{c_1x}, \lambda_{c_1y}$ ✓

3^{rd} 4^{th} equation $\Rightarrow A \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = b$

Form of equations

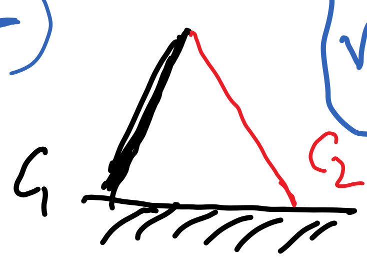
$$M \ddot{z} = B(\theta, \dot{\theta}) + J_{c_1}^T F_{c_1} + J_{c_2}^T F_{c_2}$$

$$\ddot{z} \rightarrow \{\ddot{x}, \ddot{y}, \ddot{\theta}_1, \ddot{\theta}_2\}$$

constraint forces

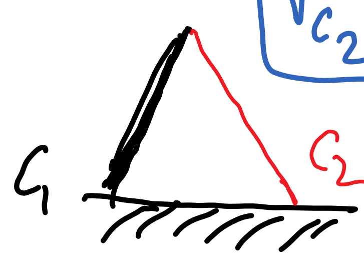
Equations for foot-strike

(−) $\boxed{v_{c_1}^- = 0}$



just before collision

(+) $\boxed{v_{c_2}^+ = 0}$

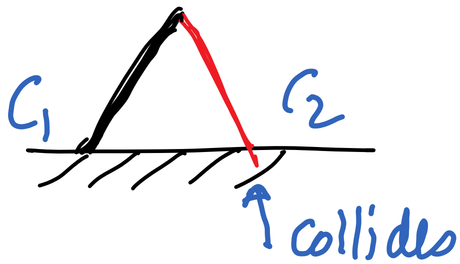


just after collision

$$\int_{t^-}^{t^+} M\ddot{z}\,dt = \cancelto{0}{\int_{t^-}^{t^+} B(\theta,\dot{\theta})\,dt} + \cancelto{0}{\int_{t^-}^{t^+} J_{c_1}^T \underbrace{F_{c_1}\,dt}_{I_{c_1}}} + \int_{t^-}^{t^+} J_{c_2}^T \underbrace{F_{c_2}\,dt}_{I_{c_2}}$$

gravity/coriolis are small during footstrike

no impulse on trailing foot C_1



$M(\theta)$ ← constant during collision

$$\left[M\dot{z}\right]_{t^-}^{t^+} = J_{c_2}^T I_{c_2}$$

$$M\dot{z}^+ - M\dot{z}^- = J_{c_2}^T I_{c_2} \quad (1)$$

— 4 equation

$z = \{x, y, \theta_1, \theta_2\}$

$\dot{z}^+ = \{ \dots$ 6 unknowns

$\{x, y, \dot{\theta}_1, \dot{\theta}_2\}^+, I_{c_2x}, I_{c_2y}$

$$V_{c_2}^{+} = J_{c_2}^{+}\,\dot{q} = 0 \qquad \text{— } (2)$$

2 equations

From (1) & (2)

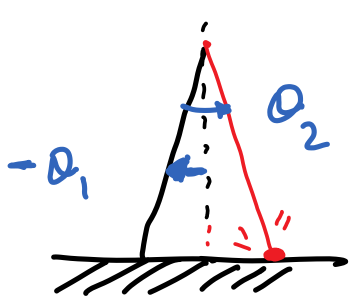
$$\begin{bmatrix} M & -J_{c_2}^{T} \\ J_{c_2} & 0 \end{bmatrix} \begin{bmatrix} \dot{\underline{z}}^{+} \\ I_{c_2} \end{bmatrix} = \begin{bmatrix} M\dot{\underline{z}}^{-} \\ 0 \end{bmatrix}$$

6 equations

6 unknowns

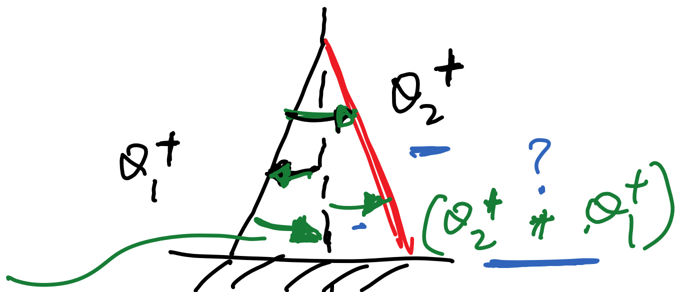
Solve for $\dot{z}^{+}$, I_{c_2}

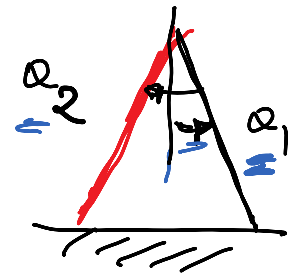
Detect collision (contact)



$$-2\theta_1 = \theta_2 \Rightarrow \boxed{2\theta_1 + \theta_2 = 0}$$

To simulate transition, we will switch the legs





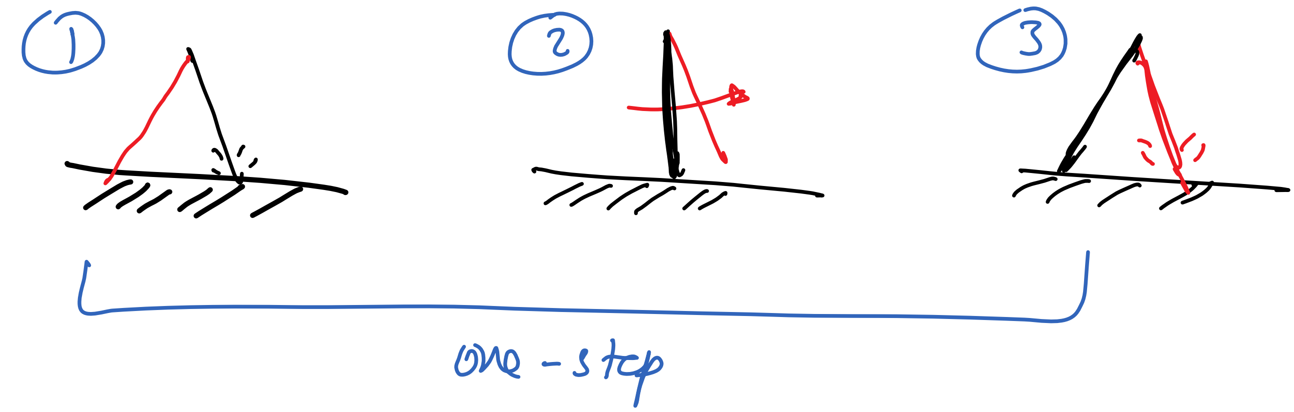
$-\theta_1^+$

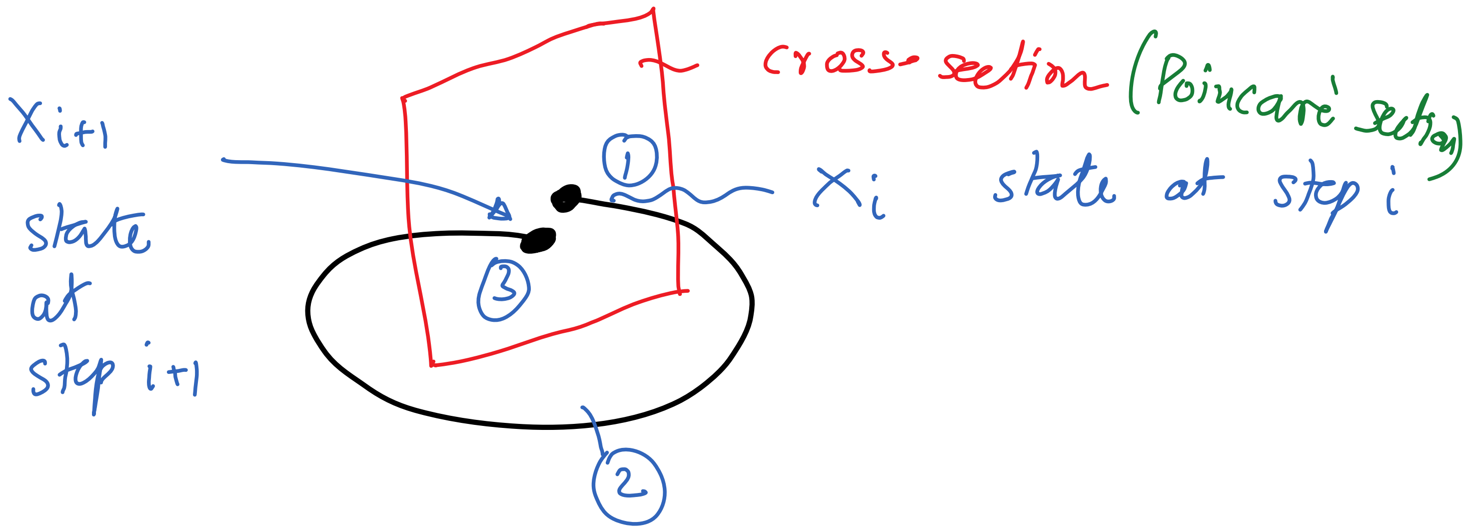
$$\boxed{\begin{aligned} -\theta_1^+ &= -\theta_1^- \\ \theta_2^+ &= \theta_2^- \end{aligned}}$$

$$\boxed{\begin{aligned} \theta_2 &= -\theta_2^+ \\ \theta_1 &= \theta_2^+ + \theta_1^+ \end{aligned}}$$

Switching angles

Analyzing walking gaits





$$x_{i+1} = F(x_i)$$

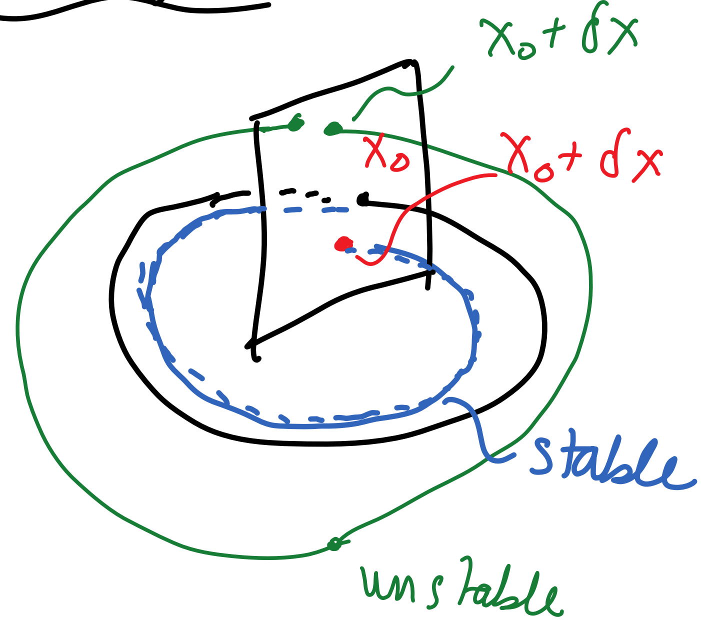
found numerically

Poincaré map

Periodic walking $\quad x_i = x_{i+1} = x_0$

$\longrightarrow \quad x_0 = F(x_0) \quad \rightsquigarrow$ Limit cycle

Stability of the limit cycle



$$J = \left.\frac{\partial F}{\partial x}\right|_{x = x_0}$$

x = state

$= \{\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2\}_{4\times1}$

$F \equiv 4\times1$

$J \equiv 4\times4$ matrix

max(eig(J)) <1 stable

>1 unstable

$=1$ neutrally stable