Jacobian (J)

Function $f = \{ f_1(q), f_2(q), f_3(q), \dots f_m(q) \}_{m \times 1}$

$$q = \{ x_1, x_2, x_3, \dots, x_n \}_{n \times 1}$$

$$J = \frac{\partial f}{\partial q}_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \cdots & & \vdots \\ \vdots & \vdots & \cdots & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example: $f = \{\overbrace{x^2+y^2}^{f_1}, \overbrace{2x+3y}^{f_2}+5\}$

$g = \{x, y\}$

$$J = \frac{\partial f}{\partial g} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 2y \\ 2 & 3 \end{bmatrix}$$

python sympy $\rightarrow$ jacobian

Application 1: Cartesian velocity

$\dot{x}, \dot{y}, \dot{z}$

Theory: $p^0 = f(q)$

joint angles

position in global frame

$J = \dfrac{\partial f}{\partial q} \longrightarrow \partial f = J \partial q$

$\dfrac{df}{dt} = J \dfrac{dq}{dt}$

$\dot{p}^0 = J \dot{q}$

linear velocity

angular velocity

$$\boxed{v = J \dot{q}}$$

Example: Double pendulum

$$v^0_{G_1} = J_{G_1}\,\dot{q} \longrightarrow \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$$

$$J_{G_1} = \frac{\partial f}{\partial q}$$

$$f = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} c_1\cos\theta_1 & 0 \\ +c_1 \sin\theta_1 & 0 \end{bmatrix}$$

$$v^0_{G_1} = \overbrace{\begin{bmatrix} c_1\cos\theta_1 & 0 \\ c_1 \sin\theta_1 & 0 \end{bmatrix}}^{J_{G_1}} \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$$

$$v^0_{G_1} = \begin{bmatrix} c_1\cos\theta_1\,\omega_1 \\ c_1 \sin\theta_1\,\omega_1 \end{bmatrix}$$

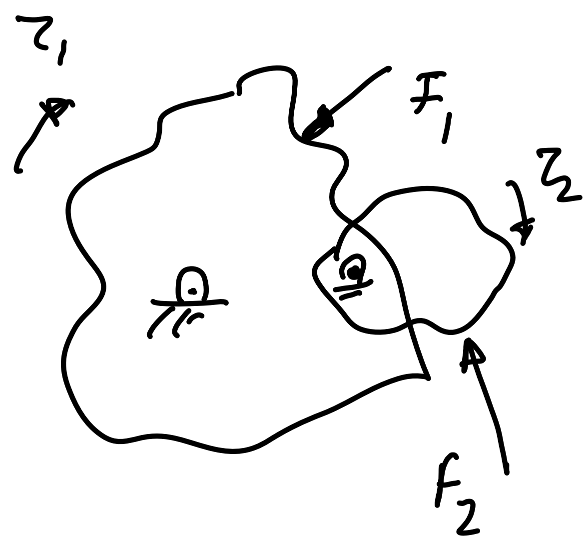
Also check $v^0_{G_2} =$ ________

$$v^0_{G_2} = J_{G_2}\,\dot{q} = \begin{bmatrix} \dfrac{\partial x^0_{G_2}}{\partial \theta_1} & \dfrac{\partial x^0_{G_2}}{\partial \theta_2} \\[2ex] \dfrac{\partial y^0_{G_1}}{\partial \theta_1} & \dfrac{\partial y^0_{G_2}}{\partial \theta_2} \end{bmatrix} \begin{array}{l} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{array} \begin{array}{l} \omega_1 \\ \omega_2 \end{array}$$

$$= \underbrace{\begin{bmatrix} \left[c_2(\cos(\theta_1+\theta_2) + l\cos\theta_1\right] & c_2\cos(\theta_1+\theta_2) \\ \left[c_2(\sin(\theta_1+\theta_2) + l\sin\theta_1\right] & c_2\sin(\theta_1+\theta_2) \end{bmatrix}}_{J_{G_2}} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Application 2: Static forces

Compute τ's given F's



In equilibrium

Virtual work

$$\underset{1\times1}{\text{Work}} = \underset{1\times2}{F^T}\ \underset{2\times1}{\delta r}$$

Force

displacement

$$\delta r = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$$\text{Work} = \tau^T \delta\theta$$

Net work = $\delta W = \tau^T \delta\theta - F^T \delta r = 0$

$$\tau^T \delta\theta = F^T \delta r$$

$$\tau^T = F^T \underbrace{\left(\frac{\delta r}{\partial \theta}\right)}_{= J}$$

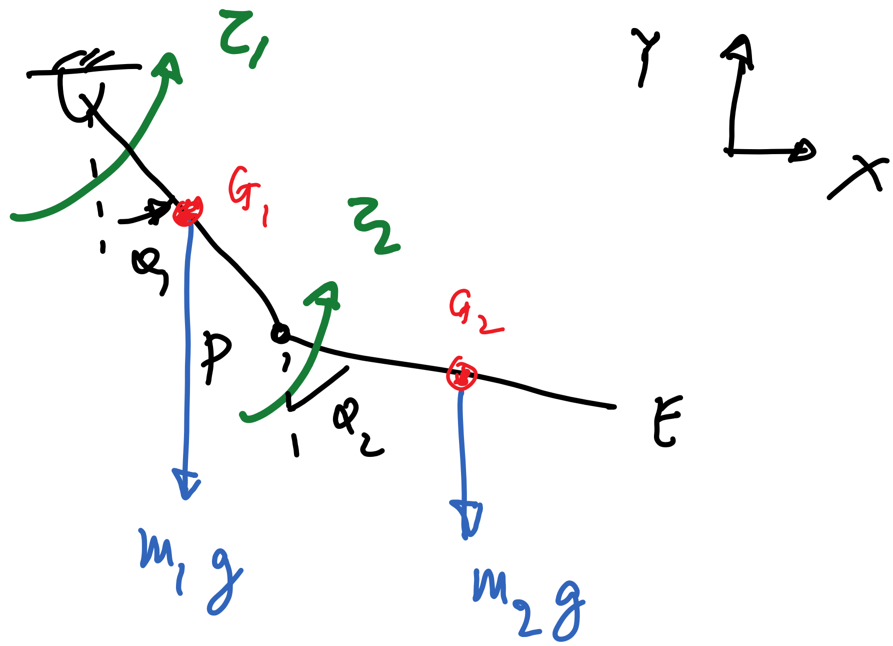
cartesian position

degrees of freedom

$$\left(\tau^T\right)^T = \left(F^T J\right)^T$$

$$\boxed{\tau = J^T F}$$

static torque



Compute τ_1, τ_2 such that the system is in equilibrium

$$\tau = J^T F$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J_{G_1}^T \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + J_{G_2}^T \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} c_1 \cos\theta_1 & c_1 \sin\theta_1 \\ 0 & 0 \end{bmatrix}}_{J_{G_1}^T} \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} +$$

$$= \underbrace{\begin{bmatrix} c_2 \cos(\theta_1+\theta_2) + l\cos\theta_1 & c_2 \sin(\theta_1+\theta_2) + l\sin\theta_1 \\ c_2 \cos(\theta_1+\theta_2) & c_2 \sin(\theta_1+\theta_2) \end{bmatrix}} \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

$J_{q_2}^T$

3) Application: Inverse kinematics

$$v = J\dot{q}$$

$$\frac{dX}{dt} = J\frac{dq}{dt} \qquad X = \{x, y\}$$

$$dX = J\,dq$$

J of the end-effector

$$dq = J^{-1}\,dX$$

$$\boxed{dq = J^{-1}(X_{ref} - X)}$$

reference

measurement

Update $Q_1 = Q_1 + dq[0]$

$Q_2 = Q_2 + dq[1]$