Symbolic derivatives / equations

Euler-lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$\mathcal{L} = T - V$$

↓

If $\mathcal{L}$ is too complex, we need to compute the derivatives using symbolics

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Symbolic derivatives

Hand calculations

$$f_0 = x^2 + 2x + 1$$

$$\frac{df_0}{dx} = 2x + 2$$

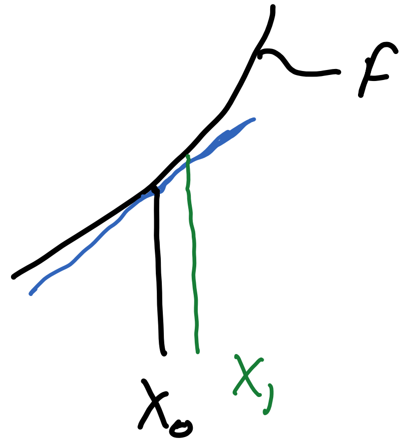
$$\left.\frac{df_0}{dx}\right|_{x=1} = 2(1)+2 = 4$$

Python

```
import sympy as sy
x = sy.symbols('x', Real=True)
f0 = x**2 + 2*x +1
df0dx = sy.diff(f0,x)
df0dx.subs(x,1)
```

Numerical derivative

$$\frac{df_0}{dx} = \frac{f_1 - f_0}{x_1 - x_0}$$



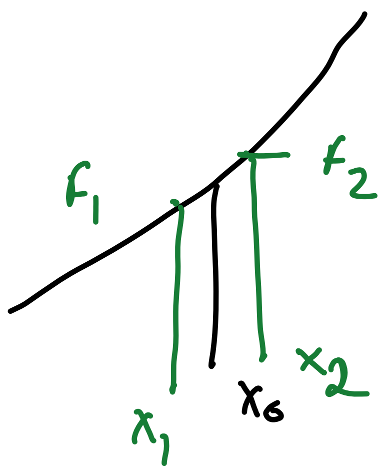
x_1 should be close to x_0

$x_1 = x_0 + 1e^{-4}$

Forward difference

$$\frac{df_0}{dx} = \frac{f_2 - f_1}{x_2 - x_1}$$

Central difference



x_1, x_2 should be close to x_0

$x_1 = x_0 - 1e^{-5}$

$x_2 = x_0 + 1e^{-5}$

Chain rule

If $f_1(x(t))$, compute $\frac{df}{dt}$

$$f_1 \longrightarrow x \longrightarrow t$$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\frac{dx}{dt} = \checkmark$$

Example

$$f_1 = \sin(x(t))$$

$$f_1 \longrightarrow x \longrightarrow t$$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\frac{dx}{dt} = \frac{d}{dx}\sin(x)\frac{dx}{dt}$$

$$\boxed{\frac{df_1}{dx} = \cos(x)\,\dot{x}}$$

If $f_2(x(t), \dot{x}(t))$ then compute $\frac{df_2}{dt}$

$$f_2 \longrightarrow x, \dot{x} \longrightarrow t$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$$f_2 = \underline{x(t)}\ \dot{x}(t)$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$$= (\dot{x})(\dot{x}) + x\,(\ddot{x})$$

$$\boxed{\frac{df_2}{dt} = \dot{x}^2 + x\ddot{x}}$$

Back to Euler-Lagrange for projectile

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\underline{\partial \dot{q}_j}}\right) - \frac{\partial \mathcal{L}}{\underline{\partial q_j}} = Q_j$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$q_j = x, y \quad Q_j = F_x, F_y$ (drag force)

$$\mathcal{L} \to x, \dot{x}, y, \dot{y} \to t$$

$\frac{\partial \mathcal{L}}{\partial \dot{q}} \to \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ ✓ No chain rule

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} \to x, \dot{x}, y, \dot{y} \to t$$

$$\frac{d}{dt}\left(\overbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}}}^{L_{\dot{x}}}\right) = \underbrace{\frac{d L_{\dot{x}}}{dx} \dot{x} + \frac{\partial L_{\dot{x}}}{d\dot{x}} \ddot{x} + \frac{\partial L_{\dot{x}}}{dy} \dot{y} + \frac{\partial L_{\dot{x}}}{d\dot{y}} \ddot{y}}_{\text{chain rule}}$$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{dx}$ (No chain rule)