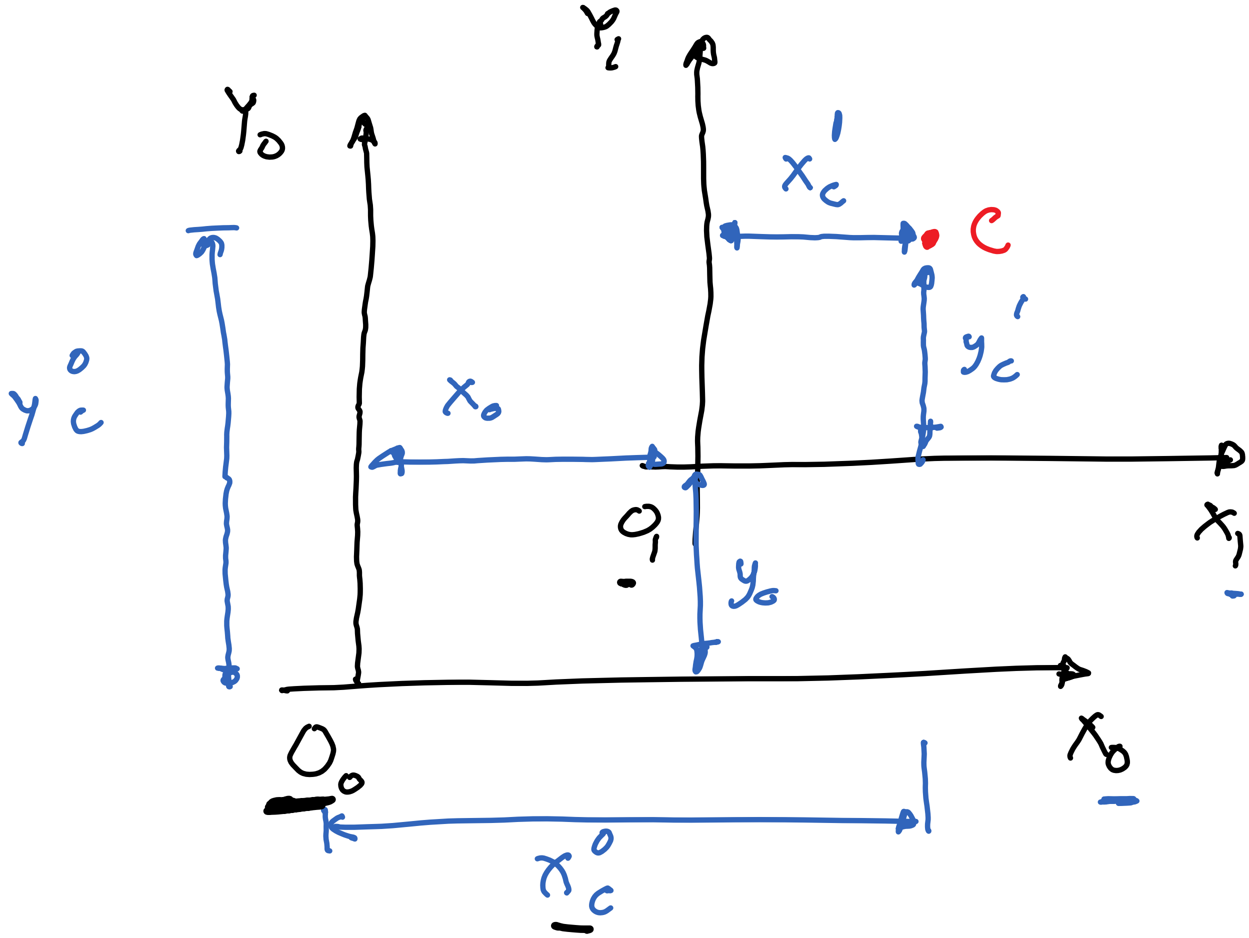
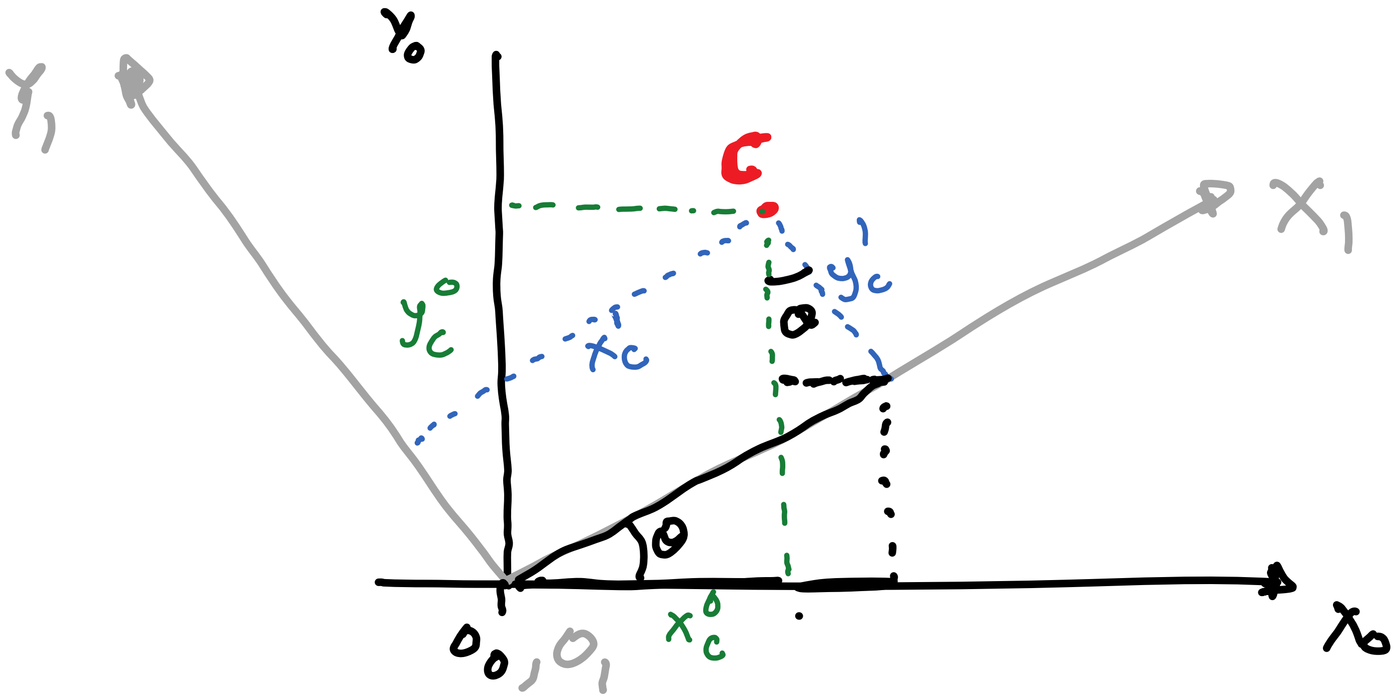
Coordinate frames

1) Translation



$$\underline{c}^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} \quad \text{or} \quad (x_c^0, y_c^0) \qquad \Bigg| \qquad \underline{O}_1^0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\underline{c}^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} \quad \text{or} \quad (x_c^1, y_c^1) \qquad \Bigg| \qquad \underline{O}_0^1 = \begin{pmatrix} -x_0 \\ -y_0 \end{pmatrix}$$



$$x_c^0 = x_c^1 \cos\theta - y_c^1 \sin\theta$$

$$y_c^0 = x_c^1 \sin\theta + y_c^1 \cos\theta$$

$$\underline{\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation matrix}} \underline{\begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}}$$

frame 0

$$\underline{C}^0 = R_1^0 \quad \underline{C}^1$$

frame 1

$$c^0 = R_1^0 \, c^1$$

$$(R_1^0)^{-1} c^0 = \underbrace{(R_1^0)^{-1} R_1^0}_{I} \, c^1$$

$$c^1 = (R_1^0)^{-1} c^0$$

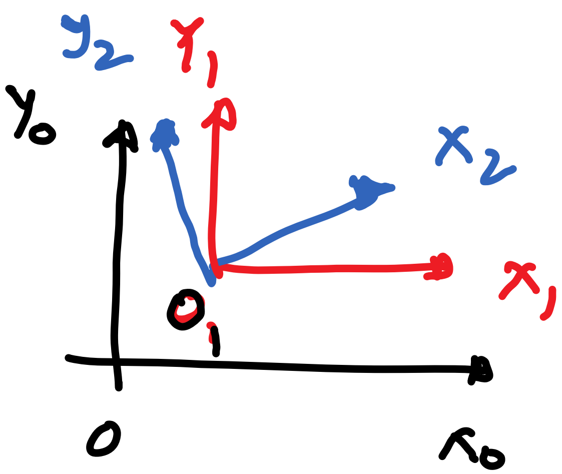
$$c^1 = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{(R_1^0)^T} c^0$$

Property of rotation matrices (R)

$$\boxed{\begin{aligned} R R^T &= I \\ R^{-1} &= R^T \end{aligned}}$$

Combining rotation and translation

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\text{Step 1}} + \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Step 2}} \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$



$$c^0 = O_1^0 + R_1^0 c^1$$

$$c^1 = O_2^1 + R_2^1 c^2$$

$$c_2^2 =$$

$$c^n = O_{n+1}^n + R_{n+1}^n c^{n+1}$$

$$C^0 = \overbrace{\left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \cdots \right.}^{\text{translation}}$$

$$+ \underbrace{\left(R_1^0 R_2^1 R_3^2 \cdots \right)}_{\text{rotation}} C^n$$

Cumbersome to remember / write

$\downarrow$

Homogenous Transformation H

$$H_i^{i-1} = \begin{bmatrix} R_{i\;\,2\times 2}^{i-1} & O_{i\;\,2\times 1}^{i-1} \\ 0_{1\times 2} & 1_{1\times 1} \end{bmatrix}_{3\times 3}$$

$$C^{i-1} = H_i^{i-1} C^i$$

is the same as

$$C^{i-1} = O_i^{i-1} + R_i^{i-1} C^i$$

$$\begin{bmatrix} C^{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C^i \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} C^i + O_i^{i-1} \\ 1 \end{bmatrix}$$

$$\boxed{C^0 = H_1^0 H_2^1 H_3^2 \cdots H_n^{n-1} C^n}$$

same as this

$$C^0 = \overbrace{\left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \cdots\right)}^{\text{translation}} + \underbrace{\left(R_1^0 R_2^1 R_3^2 \cdots\right)}\, C^n$$