Last class, Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$\mathcal{L}$ = Lagrangian $= T - V$

May be too complex to be amenable for hand derivation.

So this motivates using MATLAB for symbolic derivations.

Symbolic derivations

Hand calculation

$f_0 = x^2 + 2x + 1$

$\frac{df_0}{dx} = 2x + 2$

$\underbrace{\left.\frac{df_0}{dx}\right|_{x=1}}_{\text{symbolic derivative}} = 2(1) + 2 = \underline{4}$

MATLAB

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syms x real
f0 = x^2+2x+1
df0dx = diff(f0,x)
subs(df0dx, 1)
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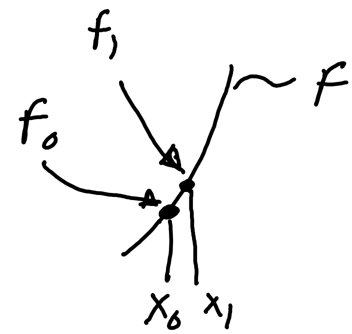
Numerical derivative

$$\left.\frac{df_0}{dx}\right|_{x_0=1}$$

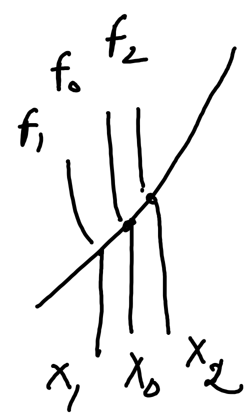


$$\boxed{\left\{ \frac{df_0}{dx} = \frac{F_1 - F_0}{x_1 - x_0} \right.}$$

x_1 shold be very close to x_0

$x_1 = x_0 + 1e^{-4}$

Forward difference



$$\boxed{\frac{df_0}{dx} = \frac{f_2 - f_1}{x_2 - x_1}}$$

x_2 & x_1 are sufficient close to x_0

Central difference ($\to$ more accurate than forward difference)

Take $x_2 = x_0 + 1e^{-5}$

$x_1 = x_0 - 1e^{-5}$

Chain rule

If $f_1(x(t))$ then find $\frac{df_1}{dt}$

$$f_1 \longrightarrow x \longrightarrow t$$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\frac{dx}{dt} \longrightarrow \text{chain rule}$$

Example $f_1 = \sin(x(t))$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\frac{dx}{dt}$$

$$= \cos(x)\,\dot{x} \qquad \left\{ \dot{x} = \frac{dx}{dt} \right\}$$

If $f_2(x(t), \dot{x}(t))$ then find $\frac{df_2}{dt}$

where $\dot{x} = \frac{dx}{dt}$

$$f_2 \longrightarrow x, \dot{x} \longrightarrow t$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$f_2 = \underline{\underline{x(t)\,\dot{x}(t)}}$ find $\frac{df_2}{dt}$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\frac{d\dot{x}}{dt}$$

$$= \dot{x}(t)\frac{dx}{dt} + x(t)\frac{d\dot{x}}{dt}$$

$$= \dot{x}\;\dot{x} + x\;\ddot{x}$$

$$\frac{df_2}{dt} = \dot{x}^2 + x\ddot{x}$$ (Answer)

Back to Euler-lagrange (Symbolic derivation of projectile equation)

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

Euler-lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

$$q = \{x, y\}$$

$$\mathcal{L} \longrightarrow x, \dot{x}, y, \dot{y} \longrightarrow t$$

$$\frac{\partial L}{\partial \dot{q}} = \mathrm{diff}(L, \dot{q}) \quad ✓ \qquad \frac{\partial L}{\partial q} = \mathrm{diff}(L, q)$$

$$= dLdqdot$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \longrightarrow \frac{d}{dx}(dLdqdot)\,\dot{x} + \frac{d}{d\dot{x}}(dLdqdot)\,\ddot{x} + \frac{d}{dy}(dLdqdot)\,\dot{y} + \frac{d}{d\dot{y}}(dLdqdot)\,\ddot{y}$$

chain rule