2D Dynamics

Newton's law (Equation of motion)

$F = ma$ $\quad$ a = acceleration

$T = I\alpha$ $\quad$ α = angular acceleration

① FBD — Free Body Diagram

② $F = ma$ or/and $T = I\alpha$

③ solve for '$\dot{a}$' or/and 'α' then integrate '$\dot{a}$' and/or 'α' to get position (x, y) or angular position (θ)

Euler-lagrange's Equations

- another way to find 'Equations of motion' without draw Free Body Diagrams.

Procedure

① Write the position of the center of mass $[x_c^0, y_c^0]$ (wrt fixed frame) & assume variable for rotation, θ_i, Then find velocities, $\dot{x}_c^0, \dot{y}_c^0, \dot{\theta}_i = \omega_i$

② Find the $\mathcal{L} = T - V$ where $\mathcal{L}$ = lagrangian

T = Kinetic energy

$$T = \frac{1}{2}\left(\sum_{i=1}^{n}\left(m_i v_i^2 + I_i \omega_i^2\right)\right)$$

where m_i = mass, I_i = inertia

v_i = linear speed, ω_i = angular speed

V = potential energy

$$V = \sum_{i=1}^{n} m_i g_i (y_c^0)_i + 0.5 \sum_{p=1}^{q} k_p (r_p - r_{p0})^2$$

g_i = gravity

k_p = spring constant

r_p, r_{p0} = spring lengths in stretched / unstretched

③ Write the equation of motion using Euler-lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j}$$ → Newton's law

This is qdot

$q_j = \{x, y, \theta\}$ $\quad Q_j = \{F_x, F_y, \tau\}$] Example

④ Integrate the Euler-Lagrange Equations to get $\dot{\theta}_i, \dot{x}, \dot{y}$ and θ_i, x, y

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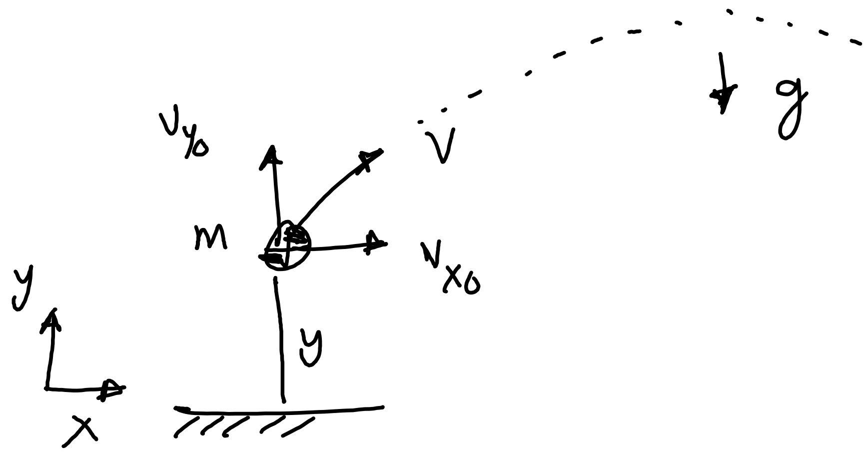
Example — Projectile motion



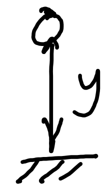
Assume there is a drag force proportional to the square of speed. Find the equation of motion & simulate & animate

$\rightarrow \quad \underline{\underline{\vec{F}_d}} = -c\, v^2 \, \hat{v} = -c\, v^2 \dfrac{\vec{v}}{|\vec{v}|} = -c\, |v|^{\cancel{2}} \dfrac{\vec{v}}{\cancel{|\vec{v}|}}$

unit vector

$$\rightarrow \begin{cases} F_x = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} \\ F_y = -c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2} \end{cases}$$

① x, y — position
$\dot{x}, \dot{y}$ → velocity



② $\mathcal{L} = T - V$

$T = 0.5\, m v^2 = 0.5\, m(\dot{x}^2 + \dot{y}^2)$

$V = mgy$

$v = \sqrt{\dot{x}^2+\dot{y}^2}$

$v^2 = \dot{x}^2+\dot{y}^2$

$$\underline{\mathcal{L}} = \underline{0.5\, m(\dot{x}^2+\dot{y}^2) - mgy}$$

③ Euler-Lagrange Equation $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$

$q_j = \{x, y\}$

$Q_j = \{f_x, f_y\} = \{-c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}, -c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}\}$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(0.5\, m(\dot{x}^2+\dot{y}^2) - mgy\right)\right) - \frac{\partial}{\partial x}\left(0.5\, m(\dot{x}^2+\dot{y}^2) - mgy\right) = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\frac{d}{dt}\left(\cancel{0.5}\, m[(\cancel{2}\dot{x}) + 0] - 0\right) - 0 = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{x} = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\boxed{\ddot{x} = -\frac{c}{m}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}} \quad — ①$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\left[0.5m(\dot{x}^2+\dot{y}^2)-mgy\right]\right)-\frac{\partial}{\partial y}\left[0.5m(\dot{x}^2+\dot{y}^2)-mgy\right]$$

$$= -c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\frac{d}{dt}\left(\cancel{0.5}m(0+\cancel{2}\dot{y})-0\right)-\left[0+0-mg(1)\right]$$

$$= -c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{y}+mg = -c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\boxed{\ddot{y} = -g-\frac{c}{m}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}} \quad - \ 2$$

4 Integrate $\ddot{x}$ & $\ddot{y}$ in 1 & 2 to get $\dot{x}, \dot{y}, x, y$ and then animate

→ Euler's, ode4 Runge-Kutta (fixed time), ode 45 (Adaptive time steps)