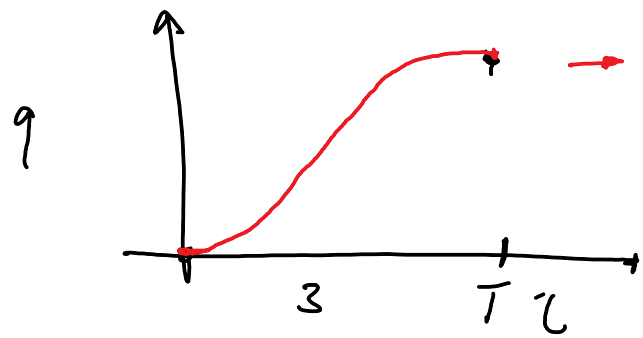
Control partitioning for trajectory tracking



$$q_{ref}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\left\{\begin{array}{l} q(0) = 0 \\ q(T) = \pi/2 \\ \dot q(0) = 0 \\ \dot q(T) = 0 \end{array}\right.$$

Goal : Joint should follow a reference trajectory.

Dynamics $\quad M\ddot q + C(q,\dot q)\dot q + G(q) = \tau \qquad$ — (1)

Controlled $\quad \tau = \hat M\left(\ddot q_{ref} - K_p(q - q_{ref}) - K_d(\dot q - \dot q_{ref})\right) + \hat G(q) + \hat C(q,\dot q)\dot q \qquad$ (2)

Analysis : when $\hat M = M \quad \hat G = G \quad \hat C = C$

Put (2) in (1)

$$M\ddot q + \cancel{C(q,\dot q)}\dot q + \cancel{G(q)} = M\left(\ddot q_{ref} - K_p(q - q_{ref}) - K_d(\dot q - \dot q_{ref})\right) + \cancel{G(q)} + \cancel{C(q,\dot q)}\dot q$$

$$M\left[(\ddot q - \ddot q_{ref}) + K_d(\dot q - \dot q_{ref}) + K_p(q - q_{ref})\right] = 0$$

$$M\left[(\ddot{q} - \ddot{q}_{ref}) + K_d(\dot{q} - \dot{q}_{ref}) + K_p(q - q_{ref})\right] = 0$$

$$q - q_{ref} = e$$

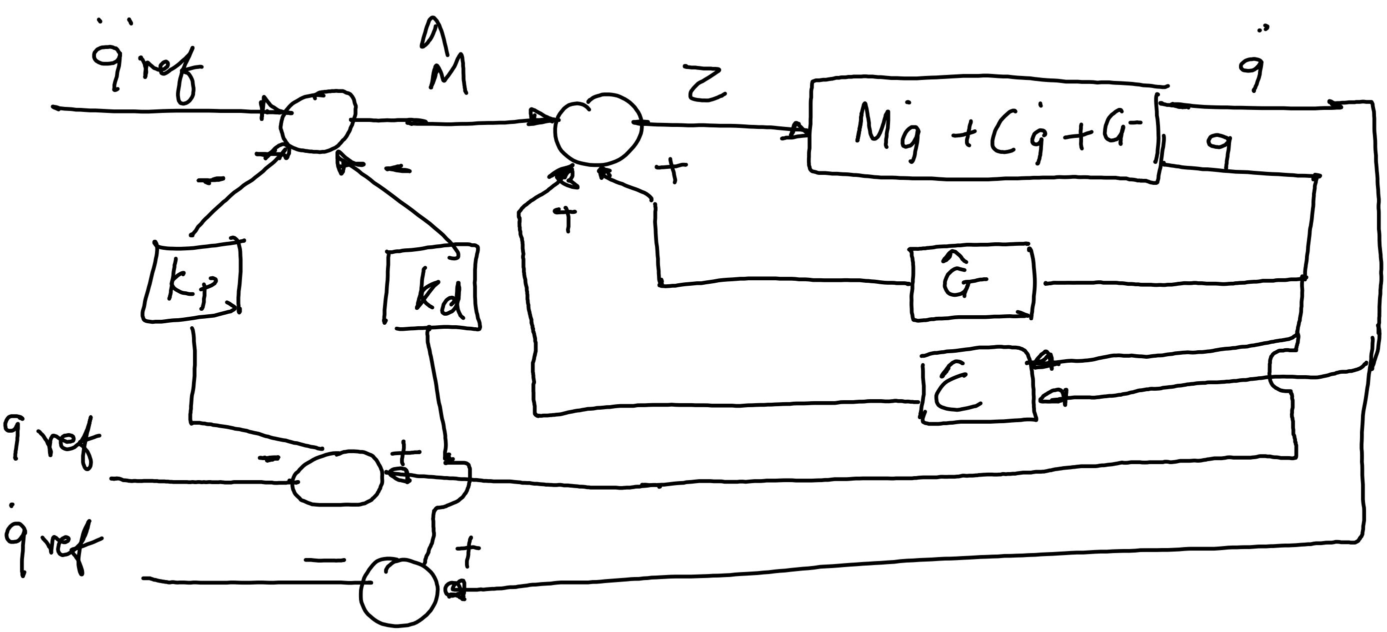
$$M\left[\ddot{e} + K_d\dot{e} + K_p e\right] = 0$$

$$M \neq 0 \Rightarrow \boxed{\ddot{e} + K_d\dot{e} + K_p e = 0}$$

similiar $\ddot{q} + K_d\dot{q} + K_p q = 0$

$K_d = 2\sqrt{K_p}$ – critically damped

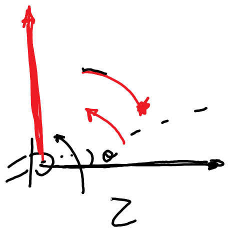
Block diagrams



$$Z = \hat{M}\left(\ddot{q}_{ref} - K_p(q - q_{ref}) - K_d(\dot{q} - \dot{q}_{ref})\right) + \hat{G}(q) + \hat{C}(q,\dot{q})\dot{q}$$

✓ ✓

① Example: Trajectory tracking of a single link pendulum



$\downarrow g$

$$\left.\begin{aligned} \theta(t=0) &= 0 \\ \theta(t=1.5) &= \pi/2 \\ \dot\theta(t=0) &= 0 \\ \dot\theta(t=1.5) &= 0 \end{aligned}\right\}$$

$\Rightarrow \quad \underline{\theta}(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$

We can find a's based on the 4 conditions

$$\left.\begin{aligned} \theta(t=1.5) &= \pi/2 \\ \theta(t=3) &= 0 \\ \dot\theta(t=1.5) &= 0 \\ \dot\theta(t=3) &= 0 \end{aligned}\right\}$$

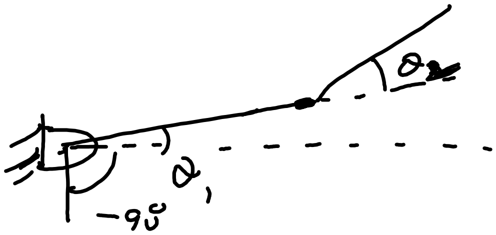
$\Rightarrow \quad \underline{\theta}(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$

find a's based on the 4 conditions

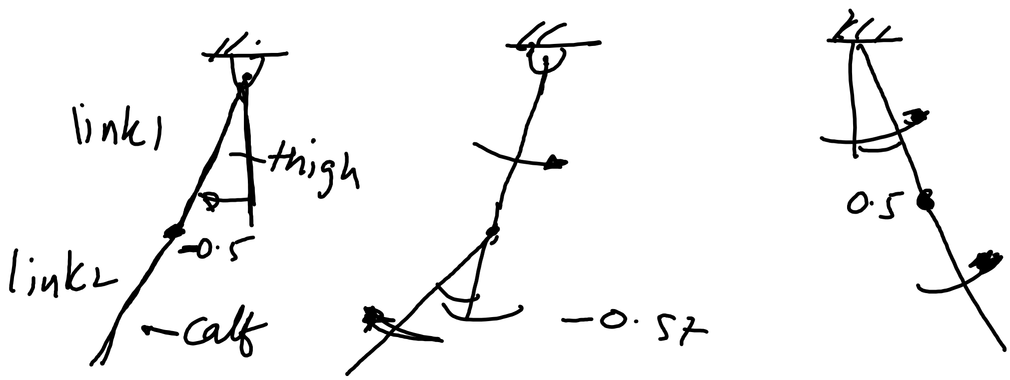
Control partitioning z

$K_d = 2\sqrt{K_p}$

② Example : double link pendulum



Trajectory



link1 : $\theta_1 \to -\frac{\pi}{2} - 0.5 \longrightarrow -\frac{\pi}{2} + 0.5$

$t \to 0 \longrightarrow 3$ sec

$\dot{\theta}_1 \to 0 \longrightarrow 0$

$$\theta_1(t) = a_0 + a_1 t + a_1 t^2 + a_2 t^3$$

& solve for a's.

link2 $\theta_2 \to 0 \longrightarrow \left(-\frac{\pi}{2} + 1\right) \longrightarrow 0$

-0.57

$t \to 0 \longrightarrow 1.5 \longrightarrow 3$

$$\theta_2(t) = \begin{cases} a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3 & 0 \le t \le 1.5 \\ a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3 & 1.5 \le t \le 3 \end{cases}$$

Cartesian-based control partitioning

→ So far q_{ref}, $\dot{q}_{ref}$, $\ddot{q}_{ref}$ q : joint positions.

→ However, we are interested in tracking end-effector position, velocity, & acceleration

$X = \{x, y, z\}$ of the end-effector

→ X_{ref}, $\dot{X}_{ref}$, $\ddot{X}_{ref}$

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① X_{ref}, $\dot{X}_{ref}$, $\ddot{X}_{ref}$ defined.

② We will find q_{ref}, $\dot{q}_{ref}$, $\ddot{q}_{ref}$ from ①

$\underline{X} = f(q)$ forward kinematics

$X_{ref} = f(q_{ref})$

$\boxed{q_{ref} = f^{-1}(X_{ref})}$ inverse kinematics

$$\dot{X} = \frac{df}{dt} = \frac{\partial f}{\partial q}\frac{\partial q}{dt} = J\dot{q}$$

$$\underline{\underline{\dot{X}_{ref}}} = J\,\underline{\underline{\dot{q}_{ref}}}$$

$$\boxed{\dot{q}_{ref} = J^{-1} \dot{x}_{ref}}$$

$$\dot{x} = J \dot{q}$$

$$\ddot{x} = J \ddot{q} + \dot{J} \dot{q}$$

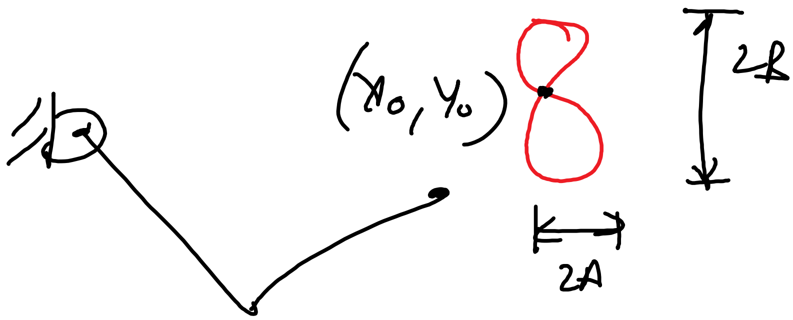
$$\ddot{q} = \ddot{x} - \dot{J} \dot{q}$$

$$\boxed{\ddot{q}_{ref} = \ddot{x}_{ref} - \dot{J} \dot{q}_{ref}}$$

③ Control partitioning

$$\tau = \hat{M} (\dot{q}_{ref} + K_p (q - \dot{q}_{ref}) + K_d (\dot{q} - \dot{q}_{ref})) + \hat{C} (q, \dot{q}) \dot{q} + \hat{G} (q)$$

Example



$$x = x_0 + A \sin 2\theta$$

$$y = y_0 + B \cos \theta$$

$\dot{x}, \ddot{x}, \dot{y}, \ddot{y} \longrightarrow$ when $\dot{\theta} = 0 \quad \dot{x}, \dot{y} = 0$

$\qquad\qquad\qquad\qquad\quad \ddot{\theta} = 0 \quad \ddot{x}, \ddot{y} = 0$

$$\left.\begin{aligned} \theta(t=0) &= 0 \\ \theta(t=T) &= 2\pi \\ \dot{\theta}(t=0) &= 0 \\ \dot{\theta}(t=T) &= 0 \\ \ddot{\theta}(t=0) &= 0 \\ \ddot{\theta}(t=T) &= 0 \end{aligned}\right\} \quad \theta = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

solve for a's based on the 6 conditions