Control of manipulators

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau$$

torque

What we did: was to simplify this equation as

$$\boxed{A X = b} \Rightarrow A \ddot{q} = b$$

Another way of writing this equation

$$A = M(q)$$
$$b = -C(q, \dot{q})\dot{q} - G(q) + \tau$$

$$M(q)\ddot{q} = -C(q,\dot{q})\dot{q} - G(q) + \tau$$

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau}$$

M — mass/inertia

C — coriolis/centripetal acceleration) torque

G — gravitational torque

τ — torque from motors

control

Simplest case: 1-D system

$$m\ddot{q} + c\dot{q} + kq = \tau$$ } spring-mass-damper

mass. damping spring term — 1D equation

Lets see what happens if $\tau = 0$

$$m\ddot{q} + c\dot{q} + kq = 0$$

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = 0 \quad — (1)$$

$$\ddot{q} + 2\xi\omega_n\dot{q} + \omega_n^2 q = 0 \quad — (2)$$

Compare (1) and (2)

$$2\xi\omega_n = \frac{c}{m} \qquad \omega_n^2 = \frac{k}{m}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$ natural frequency

$$2\xi\sqrt{\frac{k}{m}} = \frac{c}{m}$$

$$\boxed{\xi = \frac{c}{2\sqrt{mk}}}$$ damping constant

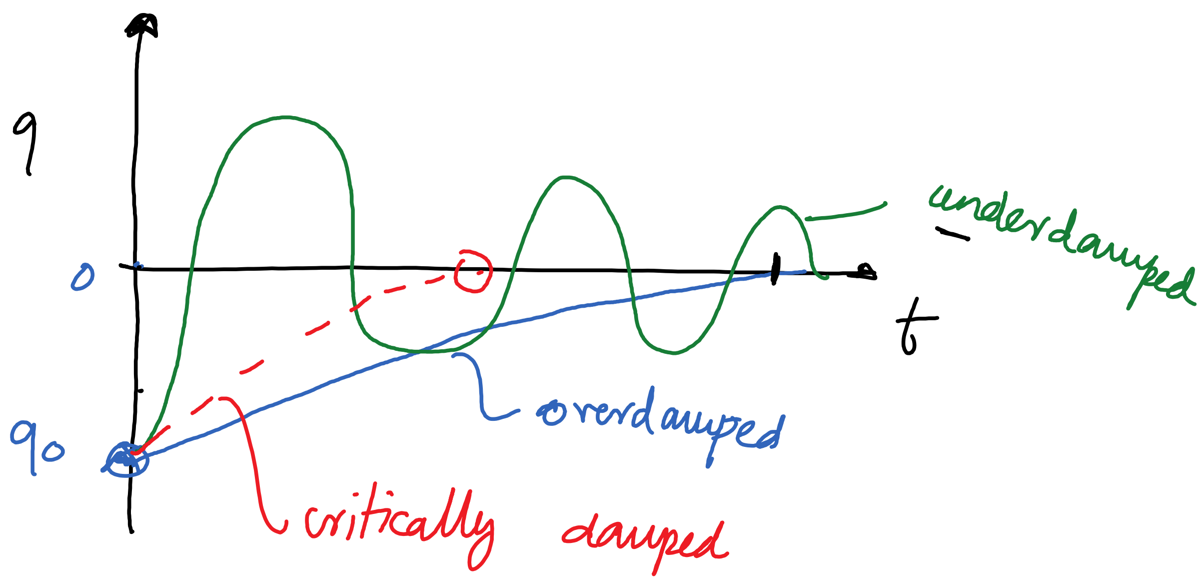
$$\rightarrow \quad c = 2\xi\sqrt{mk}$$

3 cases:

① $\xi > 1 \Rightarrow c > 2\sqrt{km}$ over damped

② $\xi = 1 \Rightarrow c = 2\sqrt{km}$ critical damped

③ $\xi < 1 \Rightarrow c < 2\sqrt{km}$ under damped.



Try to get the system to be critically damped → it goes to $q \to 0$ in the shortest time.

$$\boxed{\xi = 1 \qquad c = 2\sqrt{km}}$$

We will use z to make the system critically damped.

System

$$m\ddot{q} + c\dot{q} + kq = \tau \leftarrow \text{motor torque} \quad -(3)$$

Assume $\tau = -k_p q - k_d \dot{q}$ $\quad -(4)$

put (4) in (3)

$$m\ddot{q} + c\dot{q} + kq = -k_p q - k_d \dot{q}$$

$$m\ddot{q} + \underline{\underline{(c+k_d)}}\dot{q} + \underline{\underline{(k+k_p)}}q = 0$$

$$\ddot{q} + \frac{(c+k_d)}{m}\dot{q} + \frac{(k+k_p)q}{m} = 0$$

$$\ddot{q} + 2\zeta\omega_n \dot{q} + \omega_n^2 q = 0$$

Critically damped $c = 2\sqrt{km}$

1 equation $\longrightarrow (c+k_d) = 2\sqrt{(k+k_p)m} \quad -(5)$

2 tunable parameters
k_p, k_d

we can tune this

Square both sides

$$(c+k_d)^2 = \underline{\underline{4(k+k_p)}}m$$

$$\underline{\underline{k_d^2}} + 2c\underline{\underline{k_d}} + c^2 - 4(k+k_p)m = 0$$

$$\underline{1}\,k_d^2 + \underline{2c}\,k_d + \underline{c^2 - 4(k+k_p)m} = 0$$

$$k_d = \frac{-2c \pm \sqrt{(2c)^2 - 4(1)(c^2 - 4(k+k_p)m)}}{2(1)}$$

$$\boxed{\begin{array}{l} a'x^2 + b'x + c' = 0 \\ x = \dfrac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'} \end{array}}$$

$$k_d = -c \pm \sqrt{\cancel{c^2} - \cancel{c^2} + 4(k+k_p)m}$$

$$k_d = -c \pm 2\sqrt{(k+k_p)m}$$

Discard negative root.

$$\boxed{k_d = -c + 2\sqrt{(k+k_p)m}}$$

free parameter

to make the system be critically damped.

Illustrate in MATLAB:

$m=1,\ c=1,\ k=10 \Rightarrow \ddot{q} + \dot{q} + 10q = \tau$

$\tau = -k_p q - k_d \dot{q}$

$$k_d = -1 + 2\sqrt{(10+k_p)1}$$

$q(t=0) = 0.5$

$\dot{q}(t=0) = 0$