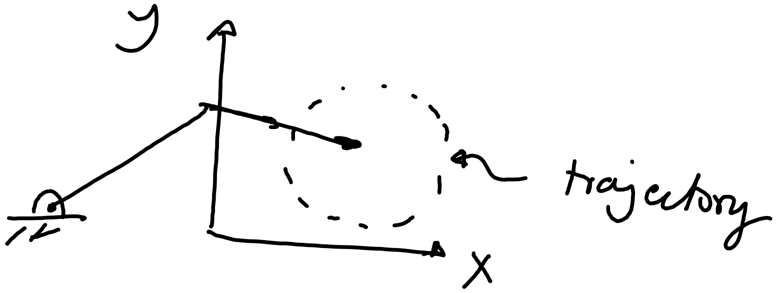
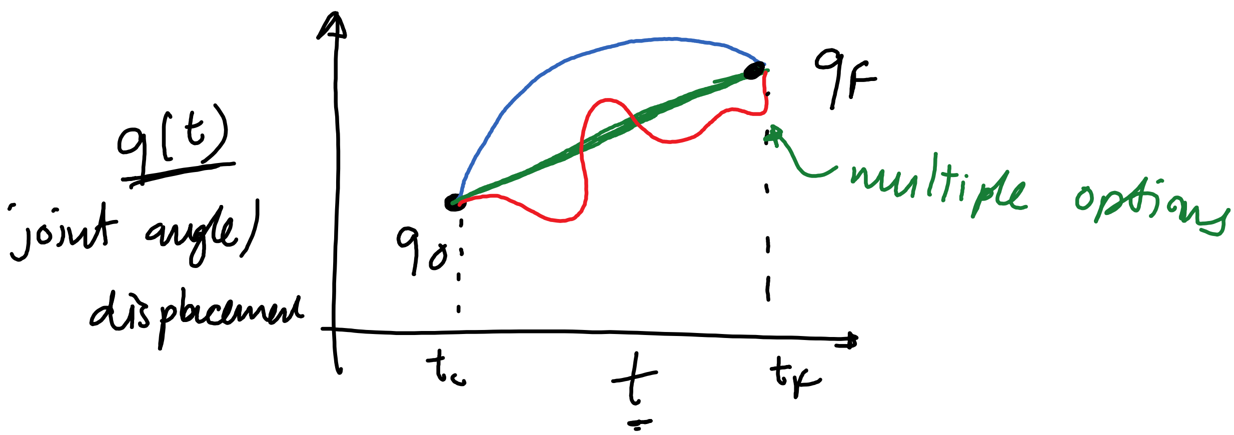
Trajectory generation

Recap



So far → static trajectory

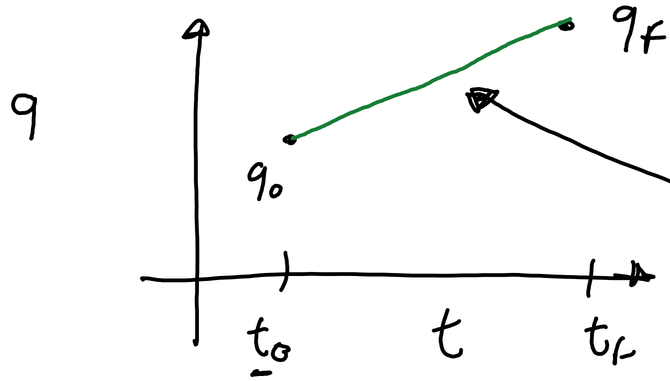
Here we will look at dynamic trajectory where there $x(t)$, $y(t)$, $z(t)$



Ways to decide which trajectory is preferred

1) Optimization : minimize energy, minimize time
2) Obstacle avoidance :
3) Avoid unnecessary jerk on the manipulator motors.

1) Lets start connecting q_0 & q_f by a straight line



Find $q(t) = ?$

$$q(t = t_0) = q_0$$
$$q(t = t_f) = q_f$$

2 conditions

$q = a_0 + a_1 t$

2 unknowns

$$q_0 = a_0 + a_1 t_0$$
$$q_f = a_0 + a_1 t_f$$

$$\begin{bmatrix} q_0 \\ q_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 \\ 1 & t_f \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$b = A X \Rightarrow$

$AX = b$

$X = A \backslash b$

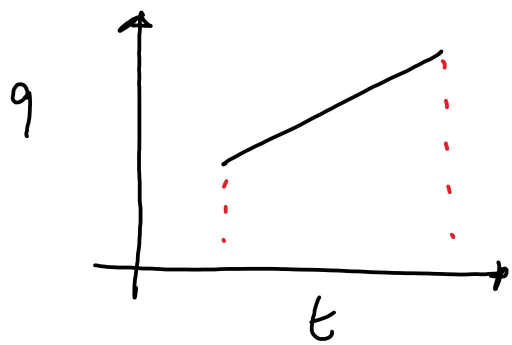
Solving

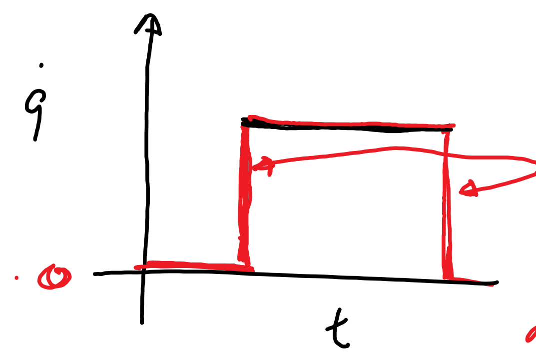
$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{(t_f - t_0)} \begin{bmatrix} q_0 t_f - q_f t_0 \\ -q_0 + q_f \end{bmatrix}$$

$\rightarrow q(t) = a_0 + a_1 t$

$$q(t) = \overbrace{\left[\frac{q_0 t_f - q_f t_0}{(t_f - t_0)}\right]}^{a_0} + \overbrace{\left[\frac{q_f - q_0}{(t_f - t_0)}\right]}^{a_1} t$$

$$\dot{q}(t) = 0 + \frac{q_f - q_0}{t_f - t_0} = \text{constant}$$





transient response can damage the motor.

How to avoid this?

$$\left.\begin{array}{llll} q(t_0) = q_0 & ; & \dot{q}(t_0) = 0 \\ q(t_f) = q_f & ; & \dot{q}(t_f) = 0 \end{array}\right\} \text{4 conditions}$$

3rd order polynomial

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \quad -(1)$$

$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \quad -(2)$$

$$0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2 \quad -(3)$$

$$0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 \quad -(4)$$

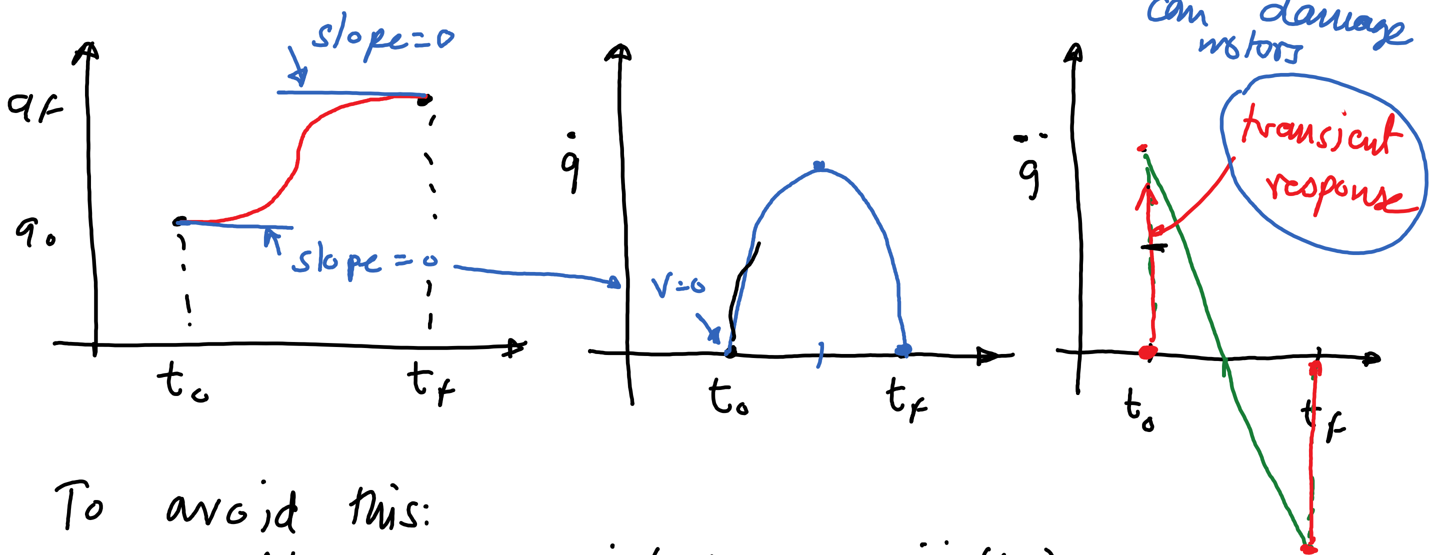
4 equations, 4 unknowns

$$\begin{bmatrix} q_0 \\ q_f \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$b = A X$$

$$X = A \backslash b \quad \text{or} \quad A^{-1} b$$

$$X = \frac{1}{(t_0 - t_f)^3} \begin{bmatrix} q_f t_0^2 (t_0 - 3t_f) + q_0 t_f^2 (3t_0 - t_f) \\ 6\, t_0 t_f (q_f - q_0) \\ 3(t_0 + t_f)(q_0 - q_f) \\ 2(q_f - q_0) \end{bmatrix}$$



To avoid this:

$$q(t_0) = q_0 \;;\quad \dot{q}(t_0) = 0 \;;\quad \ddot{q}(t_0) = 0$$
$$q(t_f) = q_f \;;\quad \dot{q}(t_f) = 0 \;;\quad \ddot{q}(t_f) = 0$$

$\underline{6}$ conditions ; $\underline{5}^{th}$ order polynomial

$$q = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

and solve for a's.

Once this is done: $q, \dot{q}, \ddot{q}$ are continuous

However $\underline{\underline{\dddot{q}}}$ is going to be discontinuous

Jerk.

$\ddddot{q} \Longrightarrow$ snap or jounce

$\overset{\cdot\cdot\cdot\cdot\cdot}{q} \Rightarrow$ crackle

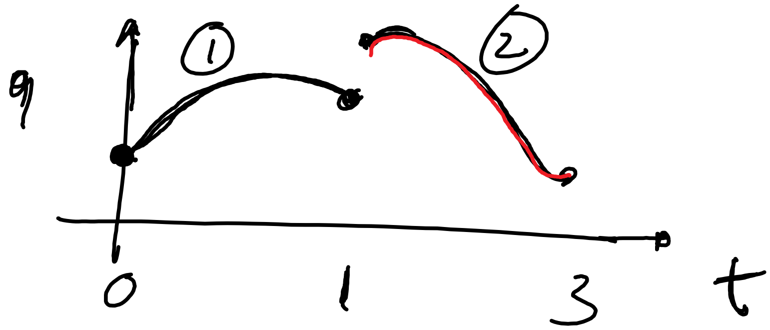
$\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{q} \Rightarrow$ pop

manipulators: $\ddot{q}$

quadcopter: $\ddddot{q}$

Example 1: Find a time based parameterization for a revolute joint of a manipulator. The joint should move from 0 to 0.5 rad from time t= 0 to t=1 sec followed by movement from 0.5 rad to 1 rad in from t=1 to t=3 secs. Also, the velocity of the joint at the start of motion (t=0) and end of motion (t=3) should be 0 and the velocity of the joint at the intermediate point (t=1) should be 0.2 rad/s. Assume two minimal order polynomials of time, one for each movement.

$$q = 0 \rightarrow 0.5 \rightarrow 1$$
$$t = 0 \rightarrow 1 \rightarrow 3$$
$$\dot{q} = 0 \rightarrow 0.2 \rightarrow 0$$



① $q_1(0) = 0$ $\quad q_1(1) = 0.5$
$\dot{q}_1(0) = 0$ $\quad \dot{q}_1(1) = 0.2$ } 4 conditions, 3rd order polynomial

② $q_2(1) = 0.5$ $\quad q_2(3) = 1$
$\dot{q}_2(1) = 0.2$ $\quad \dot{q}_2(3) = 0$ } 4 conditions, 3rd order polynomial

$$\left.\begin{aligned} q_1(t) &= a_{10} + a_{11}t^1 + a_{12}t^2 + a_{13}t^3 \\ q_2(t) &= a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3 \end{aligned}\right\} \text{8 constants}$$

$$q_1(t) = a_{10} + a_{11} t^1 + a_{12} t^2 + a_{13} t^3$$

$$q_2(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$$\dot{q}_1(t) = a_{11} + 2a_{12} t + 3a_{13} t^2$$

$$\dot{q}_2(t) = a_{21} + 2a_{22} t + 3a_{23} t^2$$

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① $q_1(0) = 0$ $\quad q_1(1) = 0.5$

$\dot{q}_1(0) = 0$ $\quad \dot{q}_1(1) = 0.2$

② $q_2(1) = 0.5$ $\quad q_2(3) = 1$

$\dot{q}_2(1) = 0.2$ $\quad \dot{q}_2(3) = 0$

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①

$$q_1(0) = a_{10} = 0 \quad \text{— (i)}$$

$$q_1(1) = a_{10} + a_{11} + a_{12} + a_{13} = 0.5 \quad \text{— (ii)}$$

$$\dot{q}_1(0) = a_{11} = 0 \quad \text{— (iii)}$$

$$\dot{q}_1(1) = a_{11} + 2a_{12} + 3a_{13} = 0.2 \quad \text{— (iv)}$$

②

$$q_2(1) = a_{20} + a_{21} + a_{22} + a_{23} = 0.5 \quad \text{— (v)}$$

$$q_2(3) = a_{20} + 3a_{21} + 9a_{22} + 27a_{23} = 1 \quad \text{— (vi)}$$

$$\dot{q}_2(1) = a_{21} + 2a_{22} + 3a_{23} = 0.2 \quad \text{— (vii)}$$

$$\dot{q}_2(3) = a_{21} + 6a_{22} + 27a_{23} = 0 \quad \text{— (viii)}$$

$$q_1(0) = a_{10} = 0 \quad \text{— (i)}$$
$$q_1(1) = a_{10} + a_{11} + a_{12} + a_{13} = 0.5 \quad \text{— (ii)}$$
$$\dot{q}_1(0) = a_{11} = 0 \quad \text{— (iii)}$$
$$\dot{q}_1(1) = a_{11} + 2a_{12} + 3a_{13} = 0.2 \quad \text{— (iv)}$$
$$q_2(1) = a_{20} + a_{21} + a_{22} + a_{23} = 0.5 \quad \text{— (v)}$$
$$q_2(3) = a_{20} + 3a_{21} + 9a_{22} + 27a_{23} = 1 \quad \text{— (vi)}$$
$$\dot{q}_2(1) = a_{21} + 2a_{22} + 3a_{23} = 0.2 \quad \text{— (vii)}$$
$$\dot{q}_2(3) = a_{21} + 6a_{22} + 27a_{23} = 0 \quad \text{— (viii)}$$

$$\begin{array}{c} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \\ \text{v} \\ \text{vi} \\ \text{vii} \\ \text{viii} \end{array}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \end{bmatrix}
\begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0.2 \\ 0.5 \\ 1 \\ 0.2 \\ 0 \end{bmatrix}$$

$$\underset{8\times 8}{A}\ \underset{8\times 1}{X} = \underset{8\times 1}{b}$$

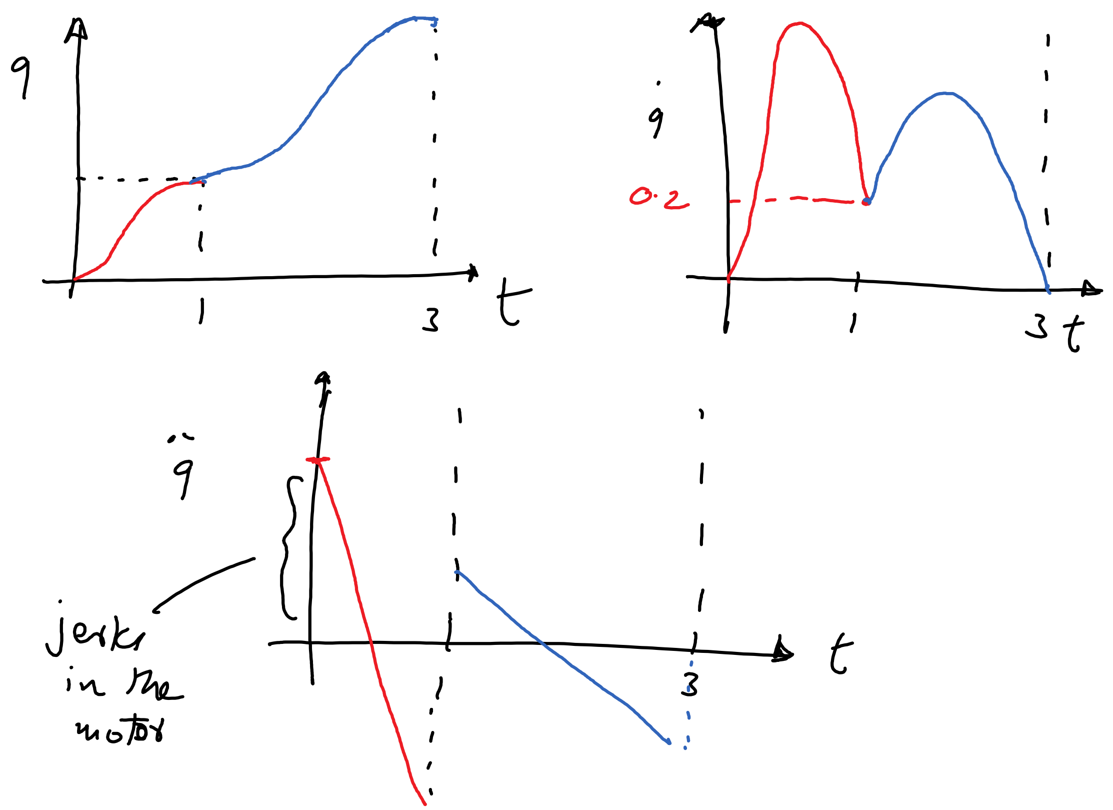
$$\underline{X} = A^{-1} b \quad \text{or} \quad A \backslash b$$

$$a_{10} = 0, \quad a_{11} = 0, \quad a_{12} = 1.3, \quad a_{13} = -0.8$$

$$a_{20} = 0.55, \quad a_{21} = -0.375, \quad a_{22} = 0.4, \quad a_{23} = -0.075$$

$q_1(t) = \ldots$

$q_2(t) = \ldots$



See MATLAB file example1.m

Example 2: Find a time based parameterization for a revolute joint of a manipulator. The joint should move from 0 to 0.5 rad from time t= 0 to t=1 sec followed by movement from 0.5 rad to 1 rad in from t=1 to t=3 secs. Also, the velocity of the joint at the start of motion (t=0) and end of motion (t=3) should be 0 and the acceleration of the joint at the intermediate point (t=1) should be continuous. Assume two minimal order polynomials of time, one for each movement.

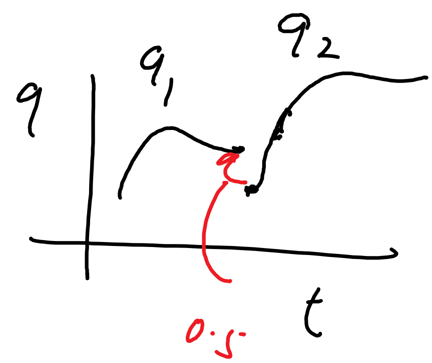
3 $\begin{cases} q_1(0) = 0, & q_1(1) = 0.5 \\ \dot{q}_1(0) = 0, & \end{cases}$

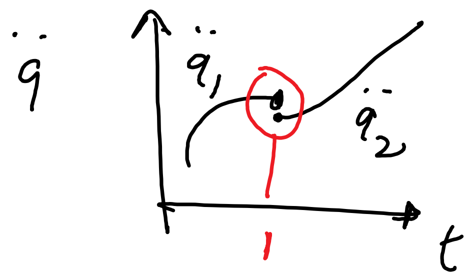
3 $\begin{cases} q_2(1) = 0.5, & q_2(3) = 1 \\ \dot{q}_2(3) = 0 & \end{cases}$

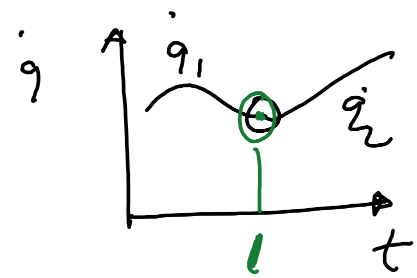
1 $\ddot{q}_1(1) = \ddot{q}_2(1)$

1 $\dot{q}_1(1) = \dot{q}_2(1)$

8 3rd order polynomial for q_1 & q_2







$$q_1(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$q_2(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$\dot{q}_1(t) =$

$\dot{q}_2(t) =$

$\ddot{q}_1(t) =$

$\ddot{q}_2(t) =$

Fill in the equations yourself.

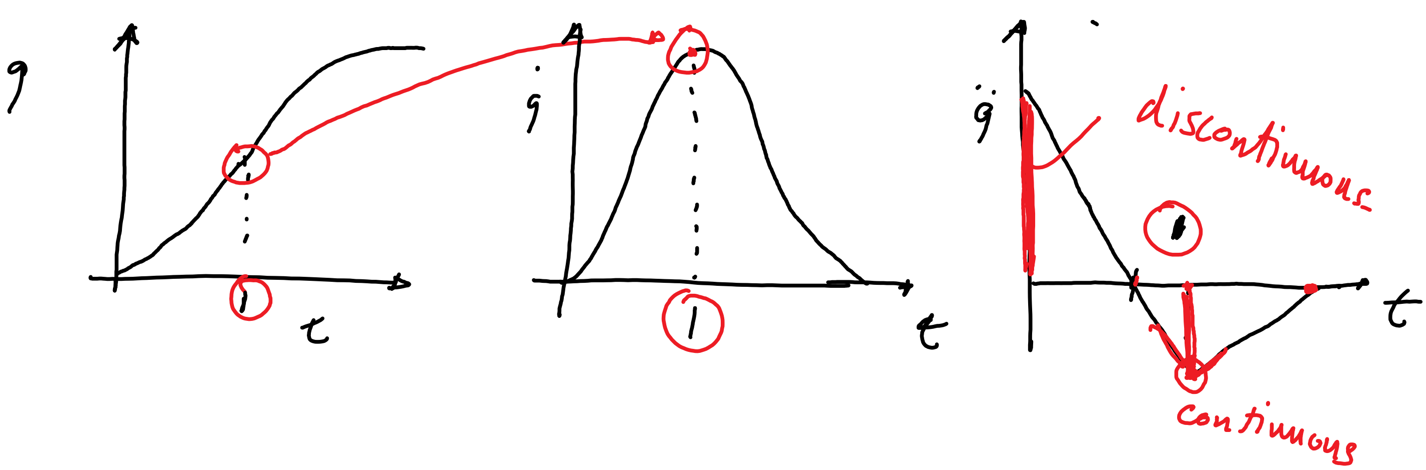
$$
\begin{array}{c}
(1) \rightarrow \\
(2) \rightarrow \\
\\
\\
\\
\\
\\
\\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\
0 & 1 & 2 & 3 & 0 & -1 & -2 & -3 \\
0 & 0 & 1 & 3 & 0 & 0 & -1 & -3
\end{bmatrix}
\begin{bmatrix}
a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0.5 \\ 0.5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

$$A \quad X \quad = b$$

$$X = A \backslash b$$

$a_{10} = 0$, $a_{11} = 0$, $a_{12} = 0.875$, $a_{13} = -0.375$

$a_{20} = -0.4063$, $a_{21} = 1.2188$, $a_{22} = -0.3438$, $a_{23} = 0.312$



See MATLAB Example 2.m