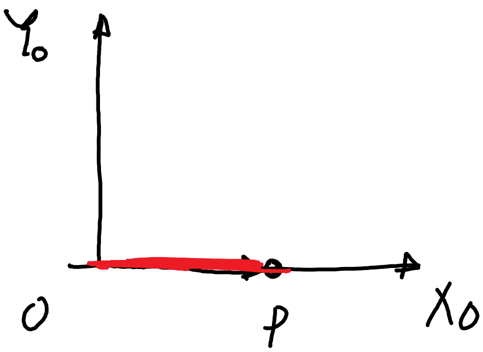
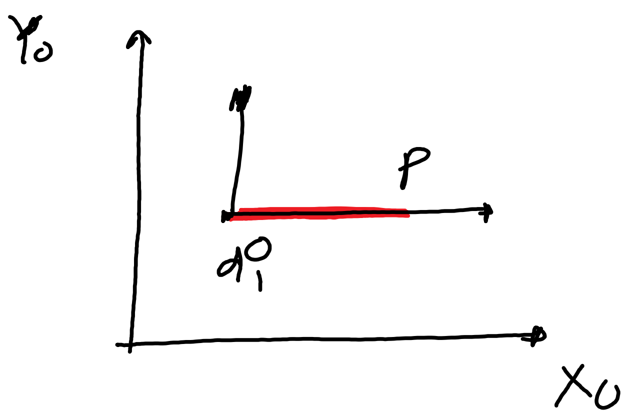
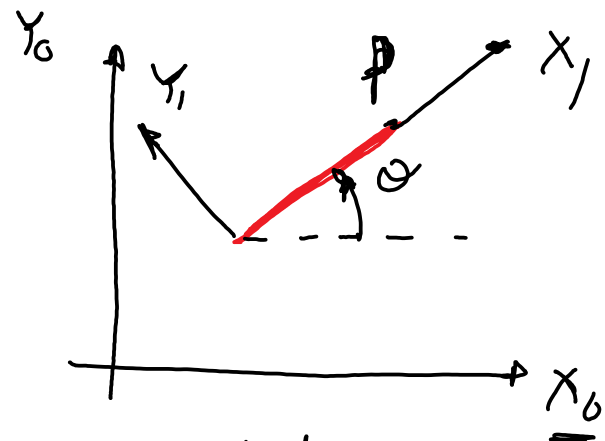
Kinematics of 3D manipulator

→ **Rigid body motion**





Translation



Rotation

We can express these transformations

$$p^0 = d_1^0 + R_1^0 p^1$$

If p^1 undergoes another translation d_2^1 & rotation R_2^1 Thus,

$$\underline{p^1} = d_2^1 + R_2^1 p^2 \quad — ②$$

$\Rightarrow$ We wrote down $p^0 = d_1^0 + R_1^0 \underline{p^1}$ — ①

Put ① in ②

from ②

$$p^0 = d_1^0 + R_1^0 \left(d_2^1 + R_2^1 \; p^2\right)$$

$$p^0 = \underbrace{\left(d_1^0 + R_1^0 d_2^1\right)}_{\text{translation}} + \underbrace{\left(R_1^0 \; R_2^1 \; p^2\right)}_{\text{rotation}}$$

We can generalize this formula for n-translations and n-rotations

$$p^0 = \underbrace{d_1^0 + R_1^0 d_2^1 + R_1^0 R_2^1 d_3^2 + \dots R_1^0 R_2^1 R_3^2 \dots R_{n-1}^{n-2} d_n^{n-1}}_{\text{translation}} + \underbrace{R_1^0 R_2^1 \dots R_n^{n-1} p^n}_{\text{rotation}}$$

This becomes unwieldy as the number of translations & rotations increase. So we will use a compact representation for the combined translation + rotation using HOMOGENOUS transformation (H)

Homogenous transformation

$$\underset{4\times4}{H} = \begin{bmatrix} R_{3\times3} & d \\ \underline{0_{1\times3}} & 1_{1\times1} \end{bmatrix}_{4\times4} \quad d: 3\times1$$

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}_{4\times4}$$

Let $\underline{P} = [p \quad 1]^T$

$$\underline{P}^0 = H_1^0 \, \underline{P}^1$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ \underline{0} & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 p^1 + d_1^0 \\ 0 + 1 \end{bmatrix} = \begin{bmatrix} R_1^0 p^1 + d_1^0 \\ 1 \end{bmatrix}$$

Compare $\Rightarrow$ $p^0 = R_1^0 \, p^1 + d_1^0$ ✓ Compare with p^0 on the previous page

$1 = 1$ ✓

$$\left.\begin{array}{l} \underline{p}^0 = H_1^0\, \underline{p}^1 \\ \underline{p}^1 = H_2^1\, \underline{p}^2 \end{array}\right\} \quad \underline{p}^0 = H_1^0 \left(H_2^1\, \underline{p}^2\right)$$

$$\underline{p}^0 = H_1^0 H_2^1\, \underline{p}^2$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{R}_2^1 & d_2^1 \\ \underline{0} & \underline{1} \end{bmatrix} \begin{bmatrix} \underline{p}^2 \\ \underline{1} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{R}_1^0 & \underline{d}_1^0 \\ \underline{0} & \underline{1} \end{bmatrix} \begin{bmatrix} R_2^1 p^2 + d_2^1 \\ \underline{1} \end{bmatrix}$$

$$= \begin{bmatrix} (R_1^0 R_2^1 p^2 + R_1^0 d_2^1) + d_1^0 \\ 0 + 1 \end{bmatrix}$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \\ 1 \end{bmatrix}$$

comparing $\quad p^0 = \underbrace{R_1^0 R_2^1 p^2}_{\text{Rotation}} + \underbrace{R_1^0 d_2^1}_{\text{translation}} + d_1^0$

comparing with $\underline{p}^0$ derived earlier, we see that we get the same expression

Conclusion: H is an easy way of keeping track of multiple rotations and translation

$$H_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3x3 3x1 1x3 1x1 4x4

$$H_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_x(a_x) = \begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3x3 3x1 1x3 1x1 4x4

$$H_y(a_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_z(a_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$