3D Angular velocity (contd.)

(1) $S(a) + S^T(a) = 0$

(2) $\vec{a} \times \vec{b} = S(a)\, b$

(3) $R\, S(a)\, R^T = S(Ra)$

S — skew symmetric matrix

$$S = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

3-2-1 Euler angle

$$r = R\, r_b$$

r - world frame

r_b - body frame

R - rotation matrix

Differentiating wrt. to time

(1) — $\dot{r} = \dot{R}\, r_b + R\, \cancel{\dot{r}_b}^{\,0}$ ~ we will come back to this

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We know that $RR^T = I$

Differentiate $\dot{R}R^T + R\dot{R}^T = 0$

$$\left.\begin{aligned} &\dot{R}R^T + \left(\left(R\dot{R}^T\right)^T\right)^T = 0 \\ &\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0 \end{aligned}\right\} (AB)^T = B^T A^T$$

where
$A = R$
$B = \dot{R}^T$

$S(a) + S^T(a) = 0 \leftarrow \underbrace{\dot{R}R^T}_{S(a)} + \left(\underbrace{\dot{R}R^T}_{S(a)}\right)^T = 0$

$S(a)^T = 0$

$$\boxed{\underbrace{S(a)}\ = \dot{R} R^T}$$ where a is some vector

Post multiply by R

$$S(a) R = \dot{R} \underbrace{R^T R}_{= I}$$

$$\boxed{\dot{R} = S(a) R} \quad \text{— (2)}$$

From (1) $\quad \dot{r} = \dot{R}\, r_b$

From (2) $\quad \dot{r} = S(a) R\, \underline{r_b} \quad$ — (3)

We know that $\quad r = R\, r_b$

$$r_b = \underline{R}^{-1} r$$

$$r_b = \underline{R}^T r \qquad \left\{ \begin{array}{l} R R^T = I \\ \underbrace{R^{-1} R}\, R^T = R^{-1} \\ R^T = R^{-1} \end{array} \right\}$$

From (3) $\quad \dot{r} = S(a) \underbrace{R\, R^T}_{= I} r$

$$\dot{r} = S(a)\, r$$

(4) — $\quad \dot{r} = a \times r \qquad \{ \vec{a} \times \vec{b} = S(a)\, b \}$

We also know that $\quad \dot{r} = \omega \times r$ — (5) $\quad$ true in 2D & 3D

From (4) and (5) $\quad a = \omega$

$$\boxed{\dot{R} R^T = \underline{S(\omega)} \quad \text{or} \quad \dot{R} = \underline{S(\omega)}\, \underline{R}}$$

$S(\omega) = ?$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

→ How is ω related to $\dot{\psi}, \dot{\theta}, \dot{\phi}$
euler angles

(*) $S(\omega) = \underline{\dot{R}} R^T$ But $\underline{R = R_z(\psi) R_y(\theta) R_x(\phi)}$

$$= \frac{d}{dt}\left(R_z R_y R_x\right) R^T$$

$$S(\omega) = \underbrace{\dot{R}_z R_y R_x R^T}_{①} + \underbrace{R_z \dot{R}_y R_x R^T}_{②} + \underbrace{R_z R_y \dot{R}_x R^T}_{③}$$

① $\underline{\dot{R}_z R_y R_x R^T} = \dot{R}_z R_y R_x \{\underline{R_z R_y R_x}\}^T$

$$= \dot{R}_z R_y R_x R_x^T R_y^T R_z^T$$

I

I

$\underline{\dot{R} = S(\omega) R}$

$$= \dot{R}_z R_z^T$$

$$= \underline{\dot{R}_z(\psi)} R_z^T(\psi) \qquad \underline{\dot{R}_z = S(\dot{\psi}\hat{k}) R_z}$$

$$= S(\dot{\psi} k) R_z R_z^T = \underline{S(\dot{\psi}\hat{k})}$$

② $\underline{\underline{R_z \dot{R}_y R_x R^T}} = R_z \dot{R}_y R_x (\underline{R_z R_y} R_x)^T$

$$= R_z \dot{R}_y \underbrace{R_x R_x^T}_{I} R_y^T R_z^T$$

$$= R_z \underline{\dot{R}_y} R_y^T R_z^T$$

$$= R_z S(\dot{\theta}\hat{j}) \underbrace{R_y R_y^T}_{I} R_z^T$$

$$= R_z S(\dot{\theta}\hat{j}) R_z^T \qquad \boxed{R S(a) R^T = S(Ra)}$$

$$= \underline{\underline{S(R_z \dot{\theta}\hat{j})}}$$

③ $\underline{\underline{R_z R_y \dot{R}_x R^T}} = R_z R_y \dot{R}_x (R_z R_y R_x)^T$

$$= R_z R_y \underline{\underline{\dot{R}_x}} (\underline{R_x^T R_y^T R_z^T})$$

$$= R_z R_y S(\dot{\phi}\hat{i}) \underbrace{R_x R_x^T}_{I} \underline{\underline{R_y^T R_z^T}}$$

$$= \underline{R_z R_y} S(\underline{\dot{\phi}}\hat{i}) (\underline{R_z R_y})^T \qquad \left\{ (AB)^T = \underline{B^T A^T} \right\}$$

$$= S(R_z R_y \dot{\phi}\hat{i})$$

$$S(\omega) = S(\dot{\psi} k) + S(R_z \dot{\theta}\hat{j}) + S(R_z R_y \dot{\phi}\hat{i})$$

$$= S(\dot{\psi}\hat{k} + R_z \dot{\theta}\hat{j} + R_z R_y \dot{\phi}\hat{i})$$

$$\boxed{\omega = \dot{\psi}\hat{k} + R_z \dot{\theta}\hat{j} + R_z R_y \dot{\phi}\hat{i}}$$