Jacobian and its application

Jacobian $\Rightarrow J$

Function $\quad F = \{ f_1(q), f_2(q), \dots f_m(q) \}_{m \times 1}$

$\qquad\qquad q = \{ x_1, x_2, \dots x_n \}_{n \times 1}$

$$J = \frac{\partial F}{\partial q}_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

Example: $\quad f = \{ x^2 + y^2, 2x + 3y + 5 \}$

$\qquad\qquad q = \{ x, y \}$

$$J = \frac{\partial f}{\partial q} = \begin{bmatrix} \frac{\partial (x^2+y^2)}{\partial x} & \frac{\partial (x^2+y^2)}{\partial y} \\ \frac{\partial (2x+3y+5)}{\partial x} & \frac{\partial (2x+3y+5)}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} 2x & 2y \\ 2 & 3 \end{bmatrix}$$

MATLAB: $\quad$ J = jacobian (f, q)

Application 1 : Cartesian velocity

Theory $\quad r^0 = f(q)$

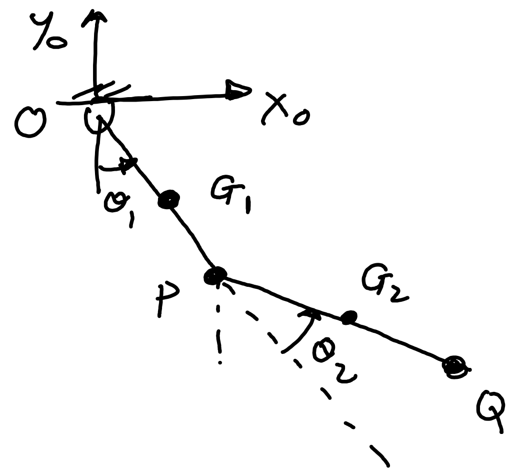
q = degrees of freedom
f = comes from rotation and translation formula

$$J = \frac{\partial f}{\partial q}$$

$$\Rightarrow \quad J \partial q = \partial f$$

$$\Rightarrow \quad J \frac{dq}{dt} = \frac{df}{dt}$$

$$\Rightarrow \quad J\dot{q} = \dot{f} = \dot{r}^0 \quad \Rightarrow \quad \boxed{\dot{r}^0 = J\dot{q}}$$



$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Find the velocity of points G_1 and G_2

$$V_{G_1} = J_{G_1} \dot{q}$$

$$g_1^0 = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} x_{G_1}^0 \\ y_{G_1}^0 \end{bmatrix}}_{f} = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix}$$

$$V_{G_1} = \left(\frac{\partial f}{\partial q}\right)\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{bmatrix} \frac{\partial x_{G_1}^0}{\partial \theta_1} & \frac{\partial x_{G_1}^0}{\partial \theta_2} \\ \frac{\partial y_{G_1}^0}{\partial \theta_1} & \frac{\partial y_{G_1}^0}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$V_{G_1} = \begin{bmatrix} c_1 \cos\theta_1 & 0 \\ c_1 \sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\boxed{V_{G_1} = \begin{bmatrix} c_1 \omega_1 \cos\theta_1 \\ c_1 \omega_1 \sin\theta_1 \end{bmatrix}}$$

$$g_2^0 = \begin{bmatrix} x_{G_2}^0 \\ y_{G_2}^0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

derive this or refer to my notes.

$$V_{G_2} = J_{G_2} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{G_2}^0}{\partial \theta_1} & \frac{\partial x_{G_2}^0}{\partial \theta_2} \\ \frac{\partial y_{G_1}^0}{\partial \theta_1} & \frac{\partial y_{G_2}^0}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$V_{Q_2} = \begin{bmatrix} \omega_1 [c_2 \cos(\theta_1+\theta_2) + l\cos\theta_1] + \omega_2 c_2 \cos(\theta_1+\theta_2) \\ \omega_1 [c_2 \sin(\theta_1+\theta_2) + l\sin\theta_1] + \omega_2 c_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

check this

$$V_P = J_P\,\dot{q} \quad ; \quad V_Q = J_Q\,\dot{q}$$

Application 2 : Static forces

Theory : Virtual work

Work done: $F^T \delta r$

$F \to [f_x, f_y]$; $\delta r \to [dx, dy]$

Work done : $\tau^T \partial\theta$

$\left.\begin{matrix} dx \\ dy \\ d\theta \end{matrix}\right\}$ virtual displacement/rotation

Total work done - $\delta W = \tau^T \partial\theta - F^T \delta r = 0$

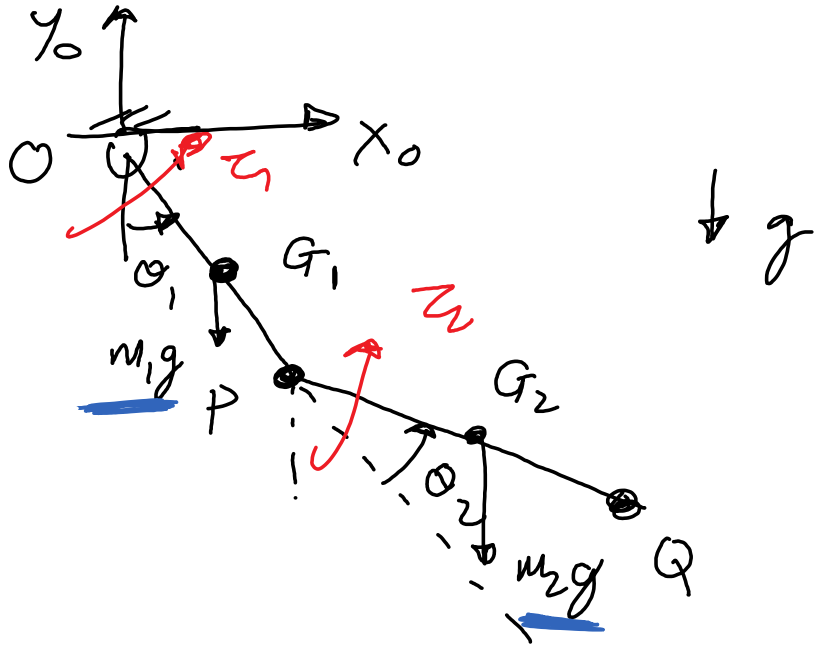
$$\Rightarrow \quad \tau^T \partial\theta = F^T \delta r$$

$$\Rightarrow \quad \tau^T = F^T \left(\frac{\delta r}{\partial\theta}\right) = \text{Jacobian} = \frac{\partial f}{\partial q}$$

$$\Rightarrow \quad \tau^T = F^T J$$

Taking transpose on both sides

$$\tau = (F^T J)^T = J^T F \quad \Rightarrow \quad \boxed{\tau = J^T F}$$



Find the torques τ_1 and τ_2 such that the double pendulum is in static equilibrium

$$\tau = J^T F$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J_{G_1}^T \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + J_{G_2}^T \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

$$J_{G_1} = \frac{\partial g_1^0}{\partial q} \quad ; \quad J_{G_2} = \frac{\partial g_2^0}{\partial q}$$

$q = \{\theta_1\ \theta_2\}$ $\quad g_1^0$ & g_2^0 (see previous section)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos\theta_1 & c_1 \sin\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + \dots$$

$$\begin{bmatrix} c_2 \cos(\theta_1+\theta_2) + l \cos\theta_1 & c_2 \sin(\theta_1+\theta_2) + l \sin\theta_1 \\ c_2 \cos(\theta_1+\theta_2) & c_2 \sin(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}$$

$$\boxed{\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -m_1 g c_1 \sin\theta_1 - m_2 g c_2 \sin(\theta_1+\theta_2) - m_2 g \ell_1 \sin\theta_1 \\ -m_2 g c_2 \sin(\theta_1+\theta_2) \end{bmatrix}}$$