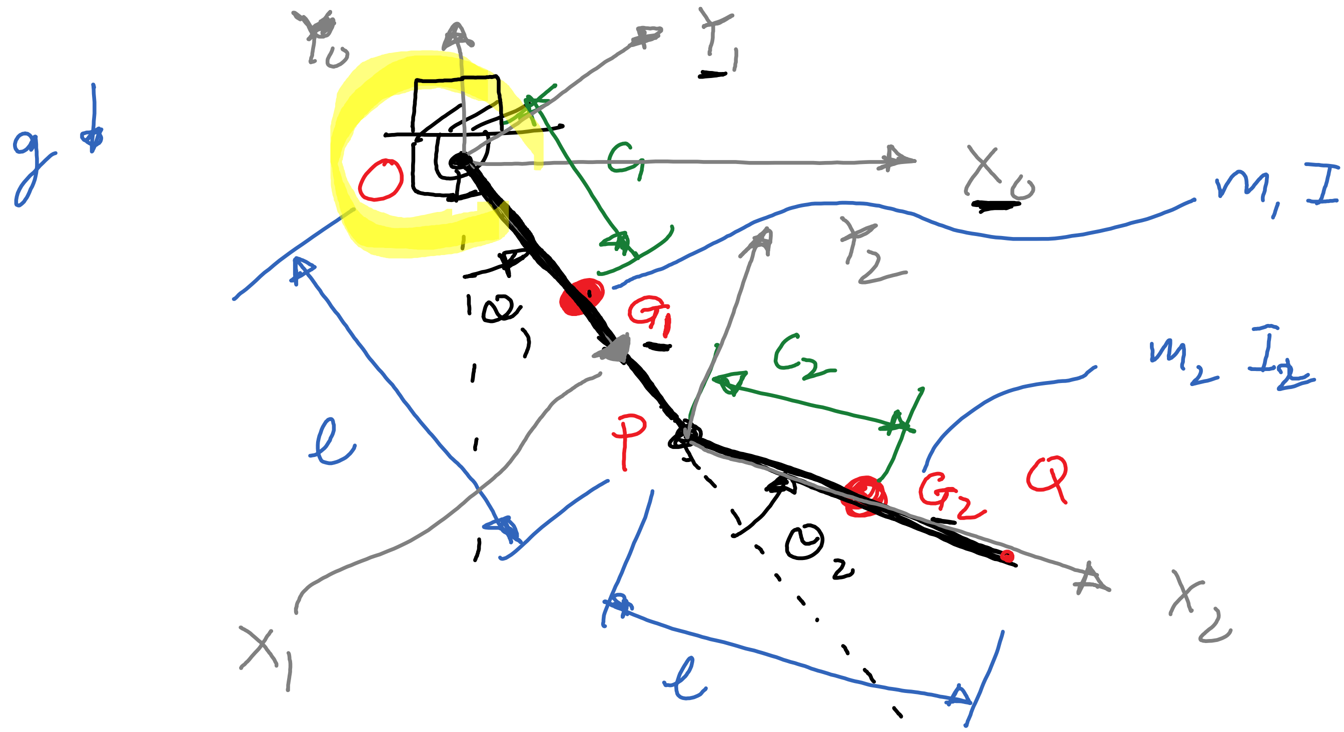
Double pendulum — simulation, derivation.



Derive equations of motion of the double pendulum

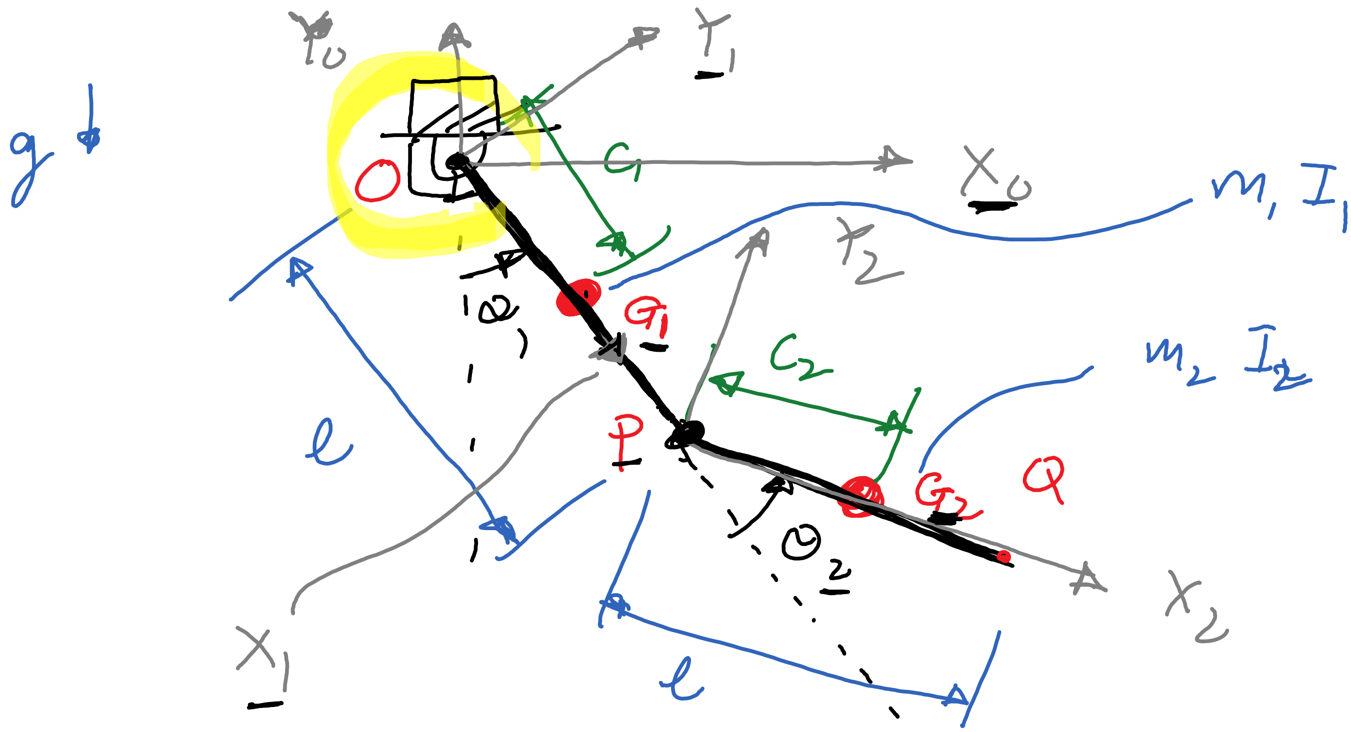
1) Position of G_1 and G_2 in frame $O X_0 Y_0$

$$g_1^0 = R_1^0 \, g_1^1$$

$$\begin{bmatrix} X_{G_1}^0 \\ Y_{G_1}^0 \end{bmatrix} = \begin{bmatrix} \cos(270+\theta_1) & -\sin(270+\theta_1) \\ \sin(270+\theta_1) & \cos(270+\theta_1) \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta_1 & \cos\theta_1 \\ -\cos\theta_1 & \sin\theta_1 \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_{G_1}^0 \\ Y_{G_1}^0 \end{bmatrix} = \begin{bmatrix} c_1 \sin\theta_1 \\ -c_1 \cos\theta_1 \end{bmatrix}$$



$$g_2^0 = p^0 + R_1^0 g_2^1 \qquad \Big\| \qquad g_2^1 = R_2^1 g_2^2$$

translation + rotation:

$$g_2^0 = p^0 + R_1^0 R_2^1 g_2^2$$

$$= R_1^0 p^1 + R_1^0 R_2^1 g_2^2$$

$$= \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 \\ -\cos\alpha_1 & \sin\alpha_1 \end{bmatrix} \begin{bmatrix} \ell \\ 0 \end{bmatrix} + \cdots$$

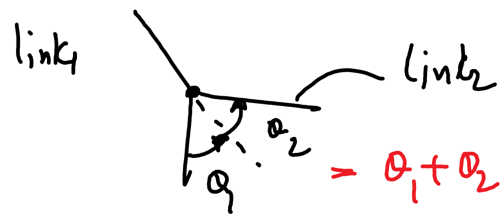
$$g_2^0 = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 \\ -\cos\alpha_1 & \sin\alpha_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} c_2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{G_2}^0 \\ y_{G_2}^0 \end{bmatrix} = \begin{bmatrix} \ell \sin\alpha_1 + c_2 \sin(\alpha_1 + \theta_2) \\ -\ell\cos\alpha_1 - c_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$V_{G_1} = \begin{bmatrix} \dot{x}^{o}_{G_1} \\ \dot{y}^{o}_{G_1} \end{bmatrix} = \begin{bmatrix} c_1 \omega_1 \cos\theta_1 \\ c_1 \omega_1 \sin\theta_1 \end{bmatrix} \qquad \boxed{\begin{aligned} \omega_1 &= \dot{\theta}_1 \\ \omega_2 &= \dot{\theta}_2 \end{aligned}}$$

$$V_{G_2} = \begin{bmatrix} \dot{x}^{o}_{G_2} \\ \dot{y}^{o}_{G_2} \end{bmatrix} = \begin{bmatrix} \omega_1 [c_2 \cos(\theta_1+\theta_2) + l\cos\theta_1] + \omega_2 c_2 \cos(\theta_1+\theta_2) \\ \omega_2 [c_2 \sin(\theta_1+\theta_2) + l\sin\theta_1] + \omega_2 c_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

2) $$T = \frac{1}{2} m_1 \left((\dot{x}^{o}_{G_1})^2 + (\dot{y}^{o}_{G_1})^2 \right) + \frac{1}{2} I_1 \omega_1^2 + \dots$$
$$\frac{1}{2} m_2 \left(\dot{x}^{o\,2}_{G_2} + (\dot{y}^{o}_{G_2})^2 \right) + \frac{1}{2} I_2 (\omega_1 + \omega_2)^2$$



$= \theta_1 + \theta_2$

ω of link 2 is $\dot{\theta}_1 + \dot{\theta}_2 = \omega_1 + \omega_2$

$$V = m_1 \underline{g}\, \underline{y^{o}_{G_1}} + m_2 g\, y^{o}_{G_2}$$

(Lagrangian) $\underline{\mathcal{L}} = T - V$

(3) Euler-lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$

$$q = \{\theta_1, \theta_2\}$$

(4) $$\left. \begin{aligned} A_{11} \ddot{\theta}_1 + A_{12} \ddot{\theta}_2 &= b_1 \\ A_{21} \ddot{\theta}_1 + A_{22} \ddot{\theta}_2 &= b_2 \end{aligned} \right\} \text{after simplifying}$$

Equations of motion

$\Downarrow$

$$\underline{\underline{A \ddot{\theta} = b}} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\ddot{\theta} = \underline{\underline{A \backslash b}} = inv(A)\, b$$

A, b are functions of $g, \theta_1, \theta_2, \omega_1, \omega_2, m_1, m_2, I_1, I_2, c_1, c_2, l$