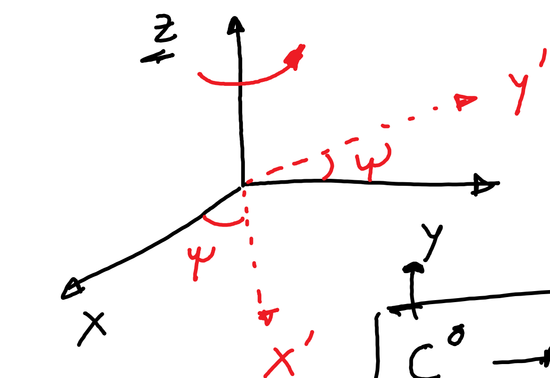
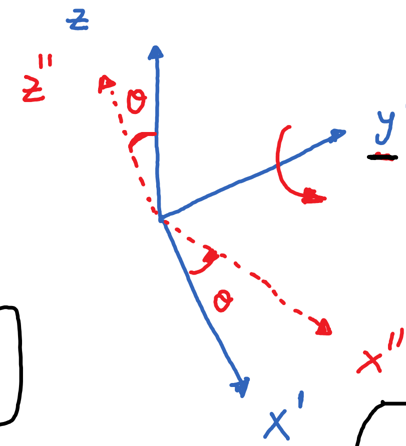
3-2-1 Euler angles

z-y-x Euler angles

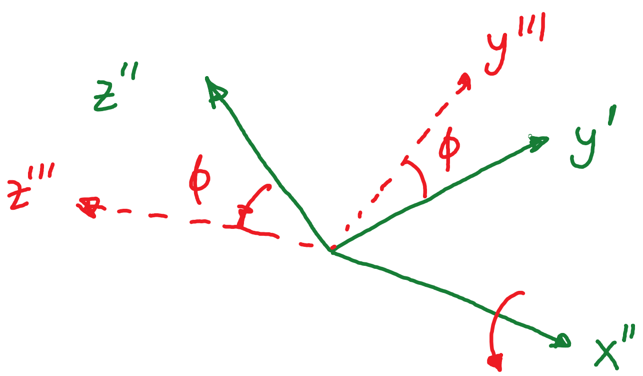
ψ, θ, ϕ



$\boxed{C^0 \longrightarrow C^1}$



$\boxed{C^1 \longrightarrow C^2}$



$\boxed{C^2 \longrightarrow C^3}$

Lets assume we had a vector in x-y-z

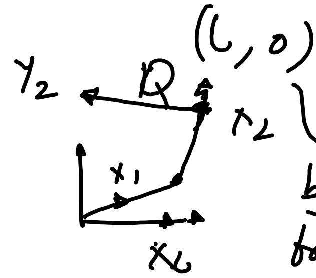
C^0 ; $x'-y'-z$ $\Rightarrow$ C^1 : $x''-y'-z''$,

C^2 : $x''-y'''-z'''$ $\longrightarrow$ C^3

$$\left.\begin{aligned} C^0 &= R_z(\psi)\, C^1 \\ C^1 &= R_y(\theta)\, C^2 \\ C^2 &= R_x(\phi)\, C^3 \end{aligned}\right\} \quad C^0 = \underbrace{R_z(\psi)\, R_y(\theta)\, R_x(\phi)}\, C^3$$

fixed frame $\qquad\qquad$ final

$$C^0 = \underline{R}\; C^3$$



body frame

$$R = R_z(\psi) R_y(\theta) R_x(\phi)$$

$$\boxed{r = R\, r_b}$$

where r_b is the position in body frame

r is the position in world frame

R is the necessary rotation matrix

$$R = \underline{R_z(\psi) R_y(\theta) R_x(\phi)}$$

$$R = \begin{bmatrix} \cos\psi \cos\theta & \cos\psi \sin\phi \sin\theta - \cos\phi \sin\psi & \sin\phi \sin\psi + \cos\phi \cos\psi \sin\theta \\ \cos\theta \sin\psi & \cos\psi \cos\phi + \sin\phi \sin\psi \sin\theta & \cos\phi \sin\psi \sin\theta - \cos\psi \sin\phi \\ -\sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta \end{bmatrix}$$

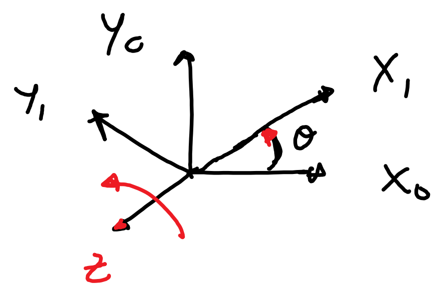
3D angular velocity

$\rightarrow$ $\boxed{\text{In 2D : } \vec{\omega}_z = \dot{\theta}\hat{k}}$ Fact1

angular speed

$\rightarrow$ vector

$\hat{k}$ unit vector along z-axis

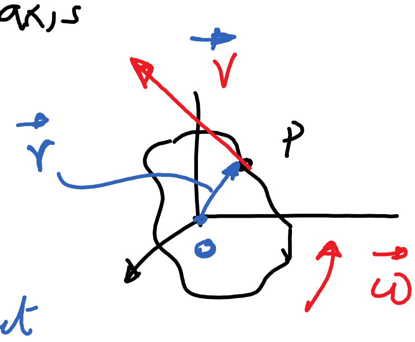


Linear speed

$\rightarrow$ $\boxed{\text{In 2D} \quad \vec{V}_p = \vec{\omega} \times \vec{r}}$

Fact2

cross product



$\rightarrow$ $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

$\rightarrow$ $\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y, z axis

$\vec{V}_p = \vec{\omega} \times \vec{r}$ (MATLAB

V= cross (ω, r)

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad \begin{bmatrix} r_1 \\ r_2 \\ r_2 \end{bmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

$$\vec{V}_p = \hat{i}(\omega_2 r_3 - \omega_3 r_2) - \hat{j}(\omega_1 r_3 - \omega_3 r_1) + \hat{k}(\omega_1 r_2 - \omega_2 r_1)$$

Fact1 : Does NOT work in 3D

i.e. $\left. \vec{\omega} \neq \dot{\phi}\hat{i} + \dot{\theta}\hat{j} + \dot{\psi}k \right\}$

How to get $\vec{\omega}$ in 3D ?

Fact2: Does Work in 3D

$\underline{\vec{V} = \vec{\omega} \times \vec{r}}$

Skew symmetrix matrix S

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$ (Definition)

$a = [a_x \quad a_y \quad a_z]$

$\uparrow$
vector

① $S(a) + S^T(a) = 0$

② $\vec{a} \times \vec{b} = S(a)\, b$ [We will prove this]

vectors x vector (matrix) (vector)

③ $R\, S(a)\, R^T = S(Ra)\, b$ 3 only works if R is a rotation matrix

R – rotation matrix

① (Proof – is in Spong's book)

② Do this calculation for a specific R matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}\,(a_y b_z - a_z b_y) - \hat{j}\,(a_x b_z - a_z b_x) + \hat{k}\,(a_x b_y - a_y b_x)$$

$$\vec{a} \times \vec{b} = \begin{array}{cl} \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} & \begin{array}{l} \leftarrow \hat{i} \text{ or } x \\ \leftarrow \hat{j} \text{ or } y \\ \leftarrow \hat{k} \text{ or } z \end{array} \end{array} \quad \text{— } 1 \quad =$$

$$S(a)\, b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \begin{bmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{bmatrix} \quad \text{— } 2 \quad =$$

$$1 = 2$$

$$\vec{a} \times \vec{b} = S(a)\, b \quad \checkmark$$