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TWO BENCHMARKS FOR OPTIMIZATION OF LEGGED ROBOTS – HYBRID SYSTEMS WITH IMPULSE EFFECTS

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Abstract. There has been an increasing trend towards using optimization to create legged robot gaits while maximizing or minimizing one or other performance metric (e.g., energy usage, speed). Because legged systems have discretely changing equations of motion, i.e., a hybrid system, such optimizations are challenging. If the optimization is incorrectly formulated it can produce infeasible or non-optimal results. There is a need to create benchmarks that can be used to test optimization softwares and techniques for legged robots. In this paper, we present two benchmarks for legged robots; passive dynamic walking and energy-optimal level ground walking. Next, we show how to use these benchmarks to validate optimization code for the given robot model, not necessarily similar to the benchmark model, by appropriate simplifications. Our hope is that such benchmarks will provide the legged robot researcher with a useful tool to not only check the optimization code but to aid in proper selection of the optimization method and/or software.

Keywords. Passive dynamic walking, Legged locomotion, Optimization benchmark, Energy-optimal control, Hybrid system.

1 Introduction

The use of optimization to create controllers for legged robots has become a popular research topic. Using optimization one can create gaits that achieve multiple objectives simultaneously. For example, creating a walking gait that minimizes the energy use but achieves a given speed and/or step length.

While there is a plethora of general purpose optimization softwares and techniques (e.g., single shooting, direct collocation), it is unclear if a given tool is able to produce the 'best' result. Although the software might be tested on generic benchmark problems (e.g., [9]) they may not work well on legged locomotion problems because of the hybrid nature of locomotion (equations of motion change with time). Sometimes the poor performance or non-convergence might be due to bad formulation of the optimization problem [16]. Thus, there is a need to create systematic benchmarks for legged locomotion that will allow users to check the validity of their optimization techniques.

In this paper, we provide two benchmarks for bipedal robots and then show how to use them to test the optimization software and technique. These benchmarks are: (1) passive dynamic walking – a legged robot with appropriate mass distribution can walk down a ramp without any control; and (2) energy-optimal walking – a point mass model of walking with extensible legs chooses an impulsive push-off from the trailing leg just before leading leg strikes the ground. Both these problems have well known solutions that serve as benchmark cases. To test the optimization code, we reduce the given robot model to be close enough to the benchmark models and run the optimization. Finally, we check if the optimization using the reduced robot model produces similar results as the benchmark case. This way we are able to validate our optimization tools. *Thus, the novelty of the paper is: (1) presentation of two benchmark optimization problems for legged robots, and (2) a technique to validate the optimization of a generic robot model against these benchmarks.*

The paper is organized as follows. In Sec. 2, we present modeling details for Ranger, the bipedal robot we want to optimize for energy usage. In Sec. 3 and Sec. 4 we provide the benchmark models of walking and show how to simplify the Ranger model to check our optimization scheme. The discussion is in Sec. 5 and conclusion is in Sec. 6.

2 Ranger robot model

Figure 1 (a) shows a photo of Cornell Ranger and Fig. 1 (b) show the dimensions, mass, and inertia parameters for the robot. The robot is on a ramp of slope γ. The feet bottoms are roughly circular arcs with radius r. The ankle joints A_1 and A_2 are offset from the center of circle by the distance d. As dictated by the geometry of circles the contact points P_1 and P_2 are always directly below the center of the circles C_1 and C_2, respectively, in level-ground walking. There is one foot configuration in which the ankle joint lies on the line joining the center of the circle and the contact point. For vertical ground forces this is a natural equilibrium position for the feet; it takes no ankle torque to hold the foot in this position. The contact point is then that part of the foot circular arc that is closest to the ankle. We call this point on the foot the 'sweet-spot'. The ankle motors are connected to the ankle joints via cables that we approximate as linear springs. The ankle motors (A_1^*, A_2^*) are actually nearly coincident with the hip H, but are separated in this diagram for clarity.

Figure 2 shows the robot joint configuration before just before heel-strike (left side and denoted by $^-$) and after heel-strike (right side and denoted by

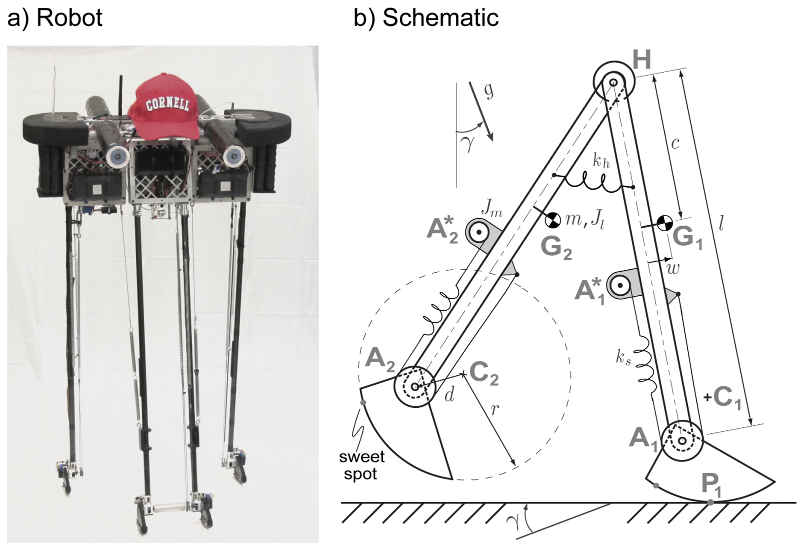

Figure 1: (a) Photo of Cornell Ranger, and (b) 2D model schematic with dimensions, mass, and inertia parameters.

$^{+}$). We swap the names of the legs during heel-strike as shown. To simplify notation, the angles are named r_i before collision and q_i after collision, where i is the joint number. The corresponding velocities are, v_i before collision and u_i after collision. We use the Fig. 1 (b) and 2 to derive the equations of motion for the benchmarks as discussed in their respective sections.

Our main motivation is to find energy-optimal walking motion for the robot Ranger. Our energy metric is the called the Cost Of Transport (COT) and is defined as the energy used per unit weight per unit distance moved. More details about the energy-optimization formulation and solution are in the thesis [2]. One question that arises when doing optimization of a complex system: *how much can we trust the optimization solutions?* One way to build trust would be to solve the optimization problem using multiple optimization softwares and multiple methods (e.g., shooting, direct collocation) and compare them. We propose a different method using legged locomotion benchmarks. *The idea is to simplify the given robot model (in this case, that of Ranger) so that it is close enough to the benchmark model. Then formulate and solve the corresponding optimization problem and check with the benchmark solution.*

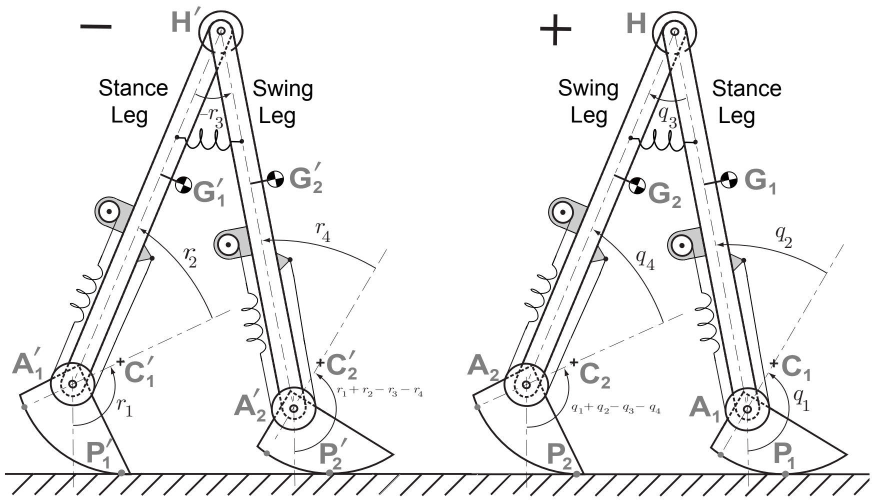

Figure 2: The robot configuration and angles before and after heel-strike are denoted by − and + respectively. Note that the legs are swapped and consequently the angles and angular velocities.

3 Benchmark one: passive dynamic walking

3.1 Overview

McGeer showed that a 2D, 2-legged robot with suitable mass distribution can walk down a shallow slope with no actuation [8]. To check our optimization against Ranger, we introduced a ramp, locked the ankles and optimized the hip motor current for energy usage. We know from previous passive-dynamics research that on a small slope this model has periodic solutions with zero hip torque and thus, with a simplified motor model, zero hip current. We also know that for the motor model the minimum conceivable energy use is zero, with zero current and thus zero torque at all times. Thus the optimal control solution should be one with zero hip current for all time, namely passive-dynamic walking.

3.2 Model simplification for 'discovery' of passive dynamic walking

For this simplification we lock the ankle joints, that is, $q_2 = q_4 =$ constant. This reduces the state space to four dimensions; the position and velocity of the stance leg q_1 and u_1 and the position and velocity of the swing leg q_3 and u_3. The equations of motion are simplified from Ranger's governing equations by doing an angular momentum balance about appropriate points (see Figs. 1 (b), 2, and see thesis [2]).

$$\begin{array}{rcc}
\text{Single stance (continuous):} & \dot{\overrightarrow{H}}_{/P_1} = \overrightarrow{M}_{/P_1} & \dot{\overrightarrow{H}}_{/H} = \overrightarrow{M}_{/H} \\
\text{Heel-strike (instantaneous):} & q_1 = r_1 - r_3 & q_3 = -r_3 \\
 & \overrightarrow{H}^{+}_{/P_1} = \overrightarrow{H}^{-}_{/P'_2} & \overrightarrow{H}^{+}_{/H} = \overrightarrow{H}^{-}_{/H'}
\end{array}$$

where $\dot{\overrightarrow{H}}_i$ is the rate of change of angular momentum about the points i, $\overrightarrow{M}_i$ is the external torque about points i, and $\overrightarrow{H}^{+}$, $\overrightarrow{H}^{-}$ are the angular moment before and after heel-strike. The locations for the single stance and heel-strike, P_1, P'_2, H, and H', are from Fig. 2. We also retained the motor model on Ranger but simplified it a bit by assuming no friction losses. This step is crucial for the optimization to discover passive dynamic walking. The electric motor model is standard one and has the following equations

$$\begin{array}{rl}
\text{Power model:} & P = I^2R + G_H K I u_3, \\
\text{Torque model:} & T_3 = G_H K I,
\end{array}$$

where the hip power is P, the hip torque is T_3, hip motor current is I, hip motor speed u_3, motor resistance is R, motor constant is K, and hip motor gear ratio is G_H. Note that this model is torque-free when the electrical power (and current) is zero. Table 1 (a) in Appendix A, gives the relevant robot and motor parameters.

3.3 Optimization problem for 'discovery' of passive dynamic walking.

We assume a walking sequence given by these phases: single stance, heel-strike, single stance, repeat. We seek a control strategy (hip motor current as a function of time) and initial conditions, that minimize the cost

$$\text{COT} = \frac{\text{Energy used per step}}{\text{Weight} \times \text{Step length}} = \int_{t=0}^{t=t_{\text{step}}} \frac{|P|dt}{M_{\text{tot}}\, g\, d_{\text{step}}} \tag{1}$$

subject to the following constraints: periodicity, namely that the state vector at the beginning of single stance should be equal to the state vector just after heel-strike; and the vertical reaction forces on the grounded legs should be positive at all times.

In Eqn. 1, M_{tot} is the total robot mass (a constant), g is gravitational constant, and d_{step} is the step length. For the optimization we replace $|P|$ with $\sqrt{P^2 + \epsilon^2}$, where $\epsilon = 0.01$. This is done to smoothen the cost function (as it is non-smooth at $P = 0$) and allows SNOPT, a gradient based method, to perform well.

Optimization parameters. The optimization parameters are: (1) the state at beginning of single stance, $x^i_{ss}(t=0) = [q_1 \ , \ q_3, \ u_1 \ , \ u_3]_{ss}(t=0)$ (4 parameters); (2) step time (t_{step}) (1 parameter); (3) the currents in the hip motor in single stance ($I(t)$). We assume piecewise linear-in-time currents. In single stance, we divide time into N intervals; $t_0, t_1, \ldots, t_N$. Here $t_{i+1} - t_i = 1/N, \ i = 0, 1, ..., N, \ t_0 = 0$ and $t_N = t_{\text{step}}$. This means there are $N+1$ unknowns for currents in single stance; $I(t=t_0), I(t=t_1), \ldots, I(t=t_N)$. Thus, there are a total of $4+1+(N+1) = N+6$ optimization parameters.

Optimization constraints. The optimization proceeds subject to various constraints on the optimization parameters and things calculated from those parameters: (1) the periodicity constraints of the state at the beginning of single stance should match the state just after heel-strike; $x^i_{ss}(t=0) = x^+_{hs}(t=t_{\text{step}})$ (4 equality constraints); (2) the transition from single stance to heel-strike takes place when the swinging leg's foot hits the ground at time $t = t_{\text{step}}$ (1 equality constraint); (3) vertical ground reaction force for the foot on the ground in single stance should be positive and this is enforced at the N+1 grid points. This gives $N+1$ inequality constraints. Thus, there are a total of 5 equality constraints and $N+1$ inequality constraints.

Method of optimization. We use SNOPT [6], a constrained optimization software based on sequential quadratic programming. SNOPT requires the user to define the cost, the optimization variables, and the optimization constraints. We create a function that takes in the optimization variables and integrates the equations of motion in single stance, applies the heel-strike condition and finally outputs the cost and the optimization constraints. Note that the motor currents are piecewise linear. To ensure that there are no discontinuities during an integration step (as the optimization needs smooth first and second derivative), we integrate from one grid point to another.

3.4 Results

Establishing passive dynamic walking benchmark. First, without any optimization, we look for passive solutions for the set of Ranger's parameters assumed here using the root-finding procedure outlined elsewhere [3, 5]. We find two such passive solutions; solution 1: $t_{\text{step}} = 0.793292$ and
$x^i_{ss}(t=0) = (q_1, u_1, q_3, u_3) = (2.997167, -0.546173, -0.288850, -0.531317)$; and
solution 2: $t_{\text{step}} = 0.936247$ and
$x^i_{ss}(t=0) = (q_1, u_1, q_3, u_3) = (2.964176, -0.652286, -0.354832, -0.027493)$.
Generally people find two periodic solutions for the passive dynamic walkers [4, 8].

Passive dynamic walking is discovered using simplified Ranger model. Next, we ran the trajectory optimization starting with initial guesses far

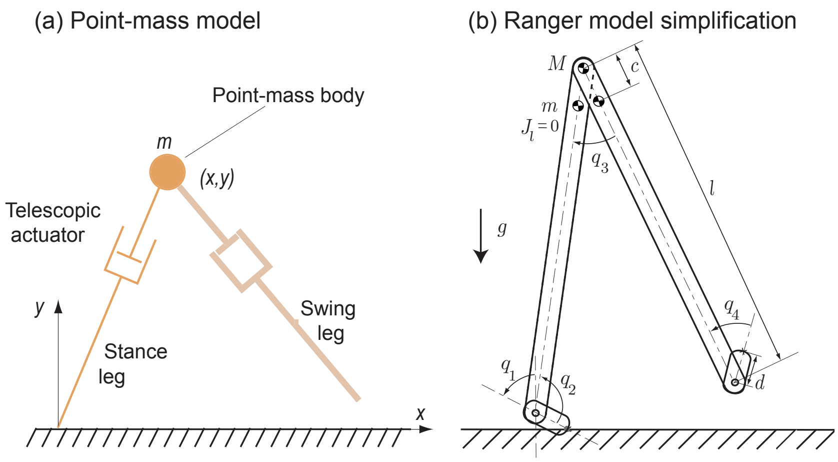

Figure 3: Benchmark two: energy-optimal level walking. (a) Point-mass model (figure source: Srinivasan [14]) (b) Ranger model simplified to a point-mass model by various special parameter values. In particular the foot radius r is set to zero, making the foot a point at a distance d from the ankle.

from the passive solutions. Each time the optimization converged to one of the above two solutions and with zero current for all time. Thus the trajectory optimization successfully discovered passive-dynamic walking.

4 Benchmark two: energy-optimal level ground walking

4.1 Overview

Srinivasan and Ruina [14, 15] present a point-mass legged locomotion model. Using energy-optimal trajectory control, they show that at low speed the model chooses a walking gait, at fast speeds the model chooses a running gait, and at intermediate speeds the model discovers a new kind of walk, which they call the pendular-run. For this benchmark we are only interested in the walking solution. In particular, the optimal strategy for walking is an impulsive push-off just before heel-strike followed by a stance phase consisting of motion as a simple inverted pendulum. For this second validation, we approximate the point-mass model by putting most of Ranger's mass on the hip, and making the legs light. We leave the foot eccentricity non-zero but make the foot radius equal to zero. In effect this makes the leg extensible, running between the hip and the infinitesimal foot (which is not inline with the leg). We then see if the optimization discovers optimum level walking with an impulsive push-off just before heel-strike.

4.2 Model simplification for 'discovery' of energy optimal level-ground walking

The original model from Srinivasan and Ruina [14, 15] is shown in Fig. 3(a) and Ranger approximation of the point-mass model is shown in Fig. 3 (b). We obtain the equations of motion from Ranger's equations of motion as follows.

$$
\begin{array}{rlr}
\text{Single stance (continuous):} & \dot{\overrightarrow{H}}_{/i} = \overrightarrow{M}_{/i}, & \text{where } i = P_1, A_1, H. \\
\text{Heel-strike (instantaneous):} & q_1 = r_1 - r_3, & q_3 = -r_3, \\
 & q_2 = r_4, & q_4 = r_2, \\
 & \overrightarrow{H}^{+}_{/P_1} = \overrightarrow{H}^{-}_{/P_2'}, & \\
 & \overrightarrow{H}^{+}_{/A_1} = \overrightarrow{H}^{-}_{/A_2'}, & \\
 & \overrightarrow{H}^{+}_{/H} = \overrightarrow{H}^{-}_{/H'} . &
\end{array}
$$

Please see Sec. 3 for definitions of the above terms. Table 1 (b) in Appendix A, gives the relevant robot and motor parameters.

4.3 Optimization problem for 'discovery' energy optimal level-ground walking

We assume a walking sequence given by these phases: single stance, heel-strike, single stance, repeat. We seek a control strategy that includes finding initial conditions in single stance, and torque in the hip and stance ankle motor as a function of time, that minimize the cost given by,

$$
\text{COT} = \frac{\text{Energy used per step}}{\text{Weight} \times \text{Step length}} = \int_{t=0}^{t=t_{\text{step}}} \frac{\{|T_2 u_2| + |T_3 u_3|\} dt}{M_{\text{tot}}\ g\ d_{\text{step}}}, \qquad (2)
$$

where ankle and hip torques are T_2 and T_3 respectively, the ankle and hip speeds are u_2 and u_3 respectively, total robot mass is M_{tot} (a constant), gravitational constant is g, and step length is d_{step}. The absolute value is not a smooth function as it has a kink at 0. So we smooth this function as by replacing $|x|$ with $\sqrt{x^2 + \epsilon^2}$, where $\epsilon = 0.01$. This make the cost function smooth and allows SNOPT, the gradient based optimization, to perform well.

We use the following constraints: periodicity requires that the state vector at the beginning of single stance should be equal to the state vector just after heel-strike; step length and step velocity is given; and the vertical reaction forces on the grounded legs should be positive at all times. Because the solution we are trying to discover has infinite forces, the numerics are helped by constraining hip and ankle motor torques to be within given bounds.

Convergence to the singular solution is inferred by the torques always using the bounds, no matter how high (see [15] for an explanation).

Numerical formulation of point-mass optimal trajectory problem

Parameters. The optimization parameters are as follows: (1) state at beginning of single stance, $x^i_{ss}(t=0) = [q_1\ ,\ q_2\ , q_3,\ u_1\ ,\ u_2\ ,\ u_3]_{ss}(t=0)$ (6 parameters); (2) the step time (t_{step}) (1 parameter); and (3) torque in the ankle motors ($T_2(t)$) and in the hip motors ($T_3(t)$). We assume piecewise linear torque profile. In single stance, we divide time into N intervals; $t_0, t_1, \ldots, t_N$. Here $t_{i+1} - t_i = 1/N,\ i = 0, 1, ..., N,\ t_0 = 0$ and $t_N = t_{\text{step}}$. This means we have $2(N+1)$ unknowns for torques in single stance; $T_j(t = t_0), T_j(t = t_1), \ldots, T_j(t = t_N)$, where $j = 2, 3$. Thus there are a total of $6 + 1 + 2(N + 1) = 2N + 9$ optimization parameters.

Constraints. The optimization constraints are as follows: (1) the state at the beginning of single stance should match the state just after heel-strike, $x^i_{ss}(t = 0) = x^+_{hs}(t = t_{\text{step}})$. (2) step velocity, v_{step}, and step length, d_{step}, are both specified; (3) transition from single stance to heel-strike takes place when the swinging leg's foot hits the ground at time $t = t_{\text{step}}$; (4) the vertical ground reaction force for the foot on the ground in single stance should be positive and are enforced at the grid points leads to $N + 1$ inequality constraints; (5) the torques in the hip and ankle motors have to be within the actuator limits and are enforced at the grid points leads to $2(N + 1)$ inequality constraints. Thus there are a total of 9 equality constraints and $3(N + 1)$ inequality constraints.

Method of optimization. This is the same as that of benchmark one (see Sec. 3).

4.4 Results

Establishing optimal walking benchmark. We present results obtained for the step velocity of $V = 0.4$ and step length of $D = 0.4$. Using the analytical solution given in Appendix B, $COT = 0.009882649139799$ at this (V, D) combination.

Optimal walking is discovered using simplified Ranger model. Using Ranger's reduced point-mass model and using a grid size $N = 12$, we calculated the COT to be 0.010113205021986. The error between our result and analytical calculations is about 2%. This error is consistent with numerical optimization results by Srinivasan [14] also taking account that this model, as opposed to the point-mass comparison, has a small, but non-zero leg swing cost. We also tried different different grid sizes, $N = 4, 8, 16, 32$

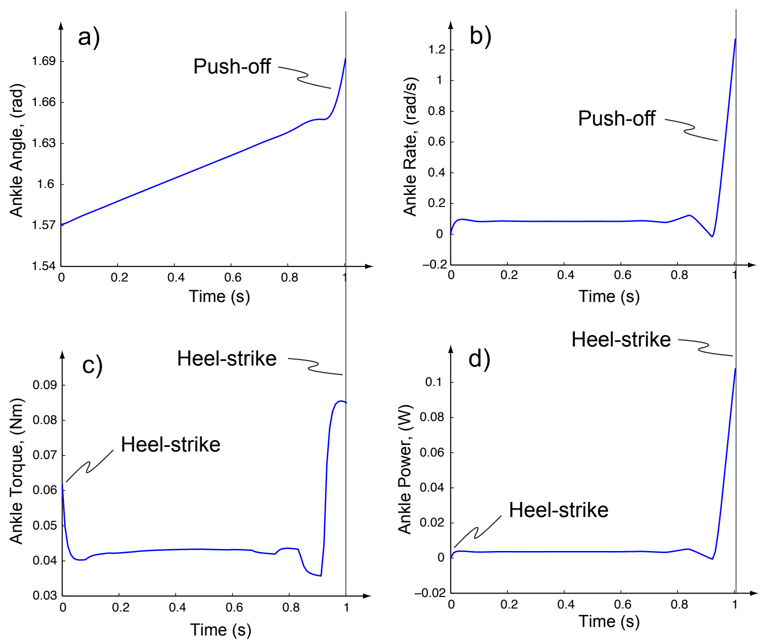

Figure 4: Ankle trajectory and controls for point-mass model limit of the Ranger model. (a) The ankle angle shows sudden lengthening at push-off; (b) The ankle rate, being near constant for the small-angle inverted-pendulum phase; (c) The ankle torque, which has no cost in this model when the ankle rate is zero, the optimizations seeming attempt to discover an impulse is shown by the spike at the right; and (d) The ankle power, which is effectively zero but for a sudden, seemingly-attempting-to-be-singular rise at push off.

and found that difference between optimization cost and true cost are consistent with those found using the point mass model (see Table 3.1, pp. 66, in Srinivasan [14]).

Figure 4 shows the trajectories for the ankle joints position and velocity, the actuator torques and the mechanical power versus time. The hip motion is low power throughout, due to the light legs. Almost all the energy for walking goes to the ankles to generate the push-off. By increasing the grid size N, we found that that the push-off becomes more pre-emptive, decreasing its duration and increasing the peak. This results suggests that the push-off tends to an impulse as the grid is made to grow infinitely big. These results are in agreement with those by Srinivasan and Ruina [15].

5 Discussion

The main result of the paper is the validation of optimization methods for a bipedal robot model by reducing it to simpler benchmark model and prescribing relevant conditions.

Optimization of gaits is becoming a popular approach for developing controllers for bipedal robots [13] as well as in biomechanics [1, 7]. Some of the possible reasons why the optimization can be challenging is due to the high dimensionality of the optimization space (e.g., humanoid robot), hybrid nature of locomotion (different equations for different phases of motion), spring-damper ground contact models which make the optimization problem stiff. One question that often arises is how much can one trust the optimization solution when analytical solutions are unknown? In this paper, we show that it possible to test the optimization of more elaborate models by reducing it to simple benchmark cases. This allows us to create trust in the solution generation by the optimization software.

Our work has limitations. We have only checked for a moderately complex model with maximum of 3 degrees of freedom. It would be interesting to try the optimization for a humanoid robot (typically with 20+ degrees of freedom) using either benchmark. We suspect that the optimization might converge to a local minimum. Another limitation is that the benchmarks are only valid for bipedal walking. For running, one can adapt the benchmarks provided in the paper by Ruina et al. [11]. For quadrupedal robots there are benchmarks for passive dynamic walking in the paper by Smith and Berkemeier [12] and Remy et al. [10]. However, we are not aware of benchmarks for problems other than those with energy-based costs.

6 Conclusion

In this paper, we provided a technique to validate optimization methods for the creation of energy-optimal bipedal walking gaits. We relied on two benchmarks whose analytical solutions are known. The key idea is to reduce the bipedal model to be close enough to the benchmark model, then run the optimization, and finally check it against the benchmark case. If the optimization produces the known solution then the optimization tool is validated. Though the current benchmark cases have limitations (e.g., the benchmarks are only for energy-optimal control, tested on relatively simple bipedal model), it is hoped that further research in this area will lead to creation of new benchmark problems that cover wide spectrum of legged locomotion cases.

(a) Passive dynamic walking

Parameter	**Value**
ℓ	0.96 m
r	0.2 m
d	0.11 m
k_h	0
J_ℓ	0.24 $\mathrm{kg\,m^2}$
m	2 kg
M	4.5 kg
w	0 m
c	0.3 m
g	10 $\mathrm{m/s^2}$
γ	0.005
G_H	66
K	0.017 N m/A
R	1.3 Ω

(b) Energy-optimal walking

Parameter	**Value**
ℓ	1 m
r	0
d	0.05 m
k_h	0
m	0.01 kg
M	1 kg
w	0
c	0.5 m
g	1 $\mathrm{m/s^2}$
γ	0

Table 1: **Reduction of Ranger to benchmark model** (a) Ranger parameter values for checking against the passive dynamic walking benchmark. (b) Ranger parameter values used for checking against the energy-optimal level walking benchmark.

A Model parameters for reduction to benchmark cases

B Energy-optimal level walking benchmark

The analytical formula for the point mass energy-optimal level ground walking is given Srinivasan [14] pp. 24-25. We briefly describe the calculation here.

The non-dimensional step velocity is V and non-dimensional step length is D. The corresponding dimensional step velocity is $v_{\text{step}} = V\sqrt{g\ell}$ and dimensional step length is $d_{\text{step}} = D\ell$, where gravitational constant is g and leg length is ℓ. The Cost Of Transport (COT) is give by the following formula.

$$\text{COT} = \frac{v_i^2 \tan^2(\alpha)}{2g d_{\text{step}}}, \tag{3}$$

where $\alpha = \sin^{-1}(D/2)$ and is one half of the angle between the legs at foot-strike, and v_i is the velocity of the point mass immediately after foot-strike. To compute v_i we first evaluate the step time using Eqn. 4 using known values of v_{step}, d_{step}, and then solving for v_i using Eqn. 5, both of which are given below.

$$t_{\text{step}} = \frac{d_{\text{step}}}{v_{\text{step}}} \tag{4}$$

$$t_{\text{step}} = 2\int_0^{\alpha} \frac{d\theta}{\dot{\theta}(t)} \quad \text{where} \quad \dot{\theta}(t) = \sqrt{\left(\frac{v_i}{\ell}\right)^2 + \left(\frac{g}{\ell}\right)(\cos\alpha - \cos\theta)}. \tag{5}$$

We used MATLAB to solve for v_i. Numerical quadrature (the function *quadl* in MATLAB) was used to find $\dot{\theta}(t)$ in Eqn. 5 and a non-linear root solver (the function *fsolve* in MATLAB) was used to solve for v_i using the value of t_{step} computed in Eqn. 4 and known values for α, g, and ℓ.

References

[1] M. Ackermann and A. V. d. Bogert. Optimality principles for model-based prediction of human gait. *Journal of biomechanics*, 43:1055–1060, 2010.

[2] P. A. Bhounsule. *A controller design framework for bipedal robots: trajectory optimization and event-based stabilization*. PhD thesis, Cornell University, 2012.

[3] P. A. Bhounsule. Numerical accuracy of two benchmark models of walking: the rimless spoked wheel and the simplest walker. *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and Algorithms*, page 137148, 2014.

[4] M. Garcia, A. Chatterjee, A. Ruina, and M. Coleman. The simplest walking model: Stability, complexity, and scaling. *ASME J. of Biomech. Eng.*, 120:281–288, 1998.

[5] M. Garcia. *Stability, scaling, and chaos in passive-dynamic gait models*. PhD thesis, Cornell University, 1999.

[6] P. Gill, W. Murray, and M. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. *SIAM Journal on Optimization*, 12(4):979–1006, 2002.

[7] A. E. Martin and J. P. Schmiedeler. Predicting human walking gaits with a simple planar model. *Journal of biomechanics*, 47(6):1416–1421, 2014.

[8] T. McGeer. Passive dynamic walking. *The International Journal of Robotics Research*, 9(2):62, 1990.

[9] H. Mittelmann. Benchmarks for optimization software. url-http://plato.la.asu.edu/bench.html, 2002. Accessed 5-April-2017.

[10] C. D. Remy, K. Buffinton, and R. Siegwart. A matlab framework for efficient gait creation. In *IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 190–196. IEEE, 2011.

[11] A. Ruina, J. Bertram, and M. Srinivasan. A collisional model of the energetic cost of support work qualitatively explains leg sequencing in walking and galloping, pseudo-elastic leg behavior in running and the walk-to-run transition. *Journal of theoretical biology*, 237(2):170–192, 2005.

[12] A. C. Smith and M. D. Berkemeier. Passive dynamic quadrupedal walking. In *IEEE International Conference on Robotics and Automation*, volume 1, pages 34–39. IEEE, 1997.

[13] K. Sreenath, H.-W. Park, I. Poulakakis, and J. W. Grizzle. A compliant hybrid zero dynamics controller for stable, efficient and fast bipedal walking on mabel. *The International Journal of Robotics Research*, 30(9):1170–1193, 2011.

[14] M. Srinivasan. *Why walk and run: energetic costs and energetic optimality in simple mechanics-based models of a bipedal animal*. PhD thesis, Cornell University, 2006.

[15] M. Srinivasan and A. Ruina. Computer optimization of a minimal biped model discovers walking and running. *Nature*, 439(7072):72–75, 2005.

[16] R. J. Vanderbei. Case studies in trajectory optimization: Trains, planes, and other pastimes. *Optimization and Engineering*, 2(2):215–243, 2001.

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